Routing on a Surface

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Introduction

Definition. Given a collection of polygons, we can pair the edges and assign an orientation to each edge. This is called a $2-$*cell embedding*

To see it is a surface, we see that around any vertex there is an orientation of edges around it that will define our surface.

Definition. We can create a *dual graph* by putting a vertex in each polygon and half edges that join to other half edges from other polygons.

Definition. For a surface Σ. we define the *Eucler Characteristic* $ech(\Sigma) := V - E + F$

Theorem. *There are two types of surfaces. Orientable or non-orintable.*

- *For an orientable surface of genus g and c cuffs, $ech(S_g^{(c)}) = 2 - 2g - c$*
- *For non-orientable surfaces with g cross caps, c cuffs, we have $ech(N_g^{(c)} = 2 - g - c$.*

Circuits

Given a circuit c on our surface, there are different types of cuts. Some will cut the surface into pieces, some will not.

Definition. A circuit is *contractioble* if one side is a dis.

Definition. For a 2-cell embed graph, two closed walks are *homotopic* if we can obtain one from the other by a series of facial exchanges or simplifications

Definition. A *facial exchange* is taking the symmetric difference of a walk with a face incident to the face

Definition. A *simplification* is taking a section of a walk that goes back on itself and removing it.

So a circuit is contractible iff it is homotopic to a trivial walk.

Lemma. *A cycle is contractible iff after cutting along it, the surface* Σ *splits into two components, one of which has* $ech(\Sigma) + 1$*.*

Question. *Given* Σ *with Euler genus* g*, how many cycles that are pairwise non-homotopic and disjoint could there be?*

Theorem. *Suppose* $C_1, \ldots, C_k$ *are non-contractible cycles, pairwise disjoint and non-homotopic on a surface* $\Sigma = S_g$ *or* N_g *(ie. no cuffs). Then:*

$$k \leq \begin{cases} g & (g \leq 1) \\ 3g - 3 & (g \geq 2) \end{cases}$$

Proof. Suppose first that Σ is orientable. Cut along each C_i. We get some surface and each C_i gives rise to 2 cuffs. So we have $ech(\Sigma^{(2k)} = 2 - 2g$. We have several components of our surface. If we add a charge of $\frac{1}{3}$ to each of our cuffs, we see that any component must have augmented weight that is non-positive. We get

$$2 - 2g = \sum_i ech(\Sigma_i) \leq -\frac{2k}{3} \Rightarrow k \leq 3g - 3$$

A similar argument works for the non-orintable case. □

Definition. A $k-$*system* of cycles $C_i, \ldots, C_1$ is a set of cycles so that no two intersect more than k times.

Theorem. *For every* Σ *and each* k*, there exists a constant* $q = q(\Sigma, k)$ *such that every* $k-$*system of pairwise non-homotopic cycles on* Σ *contains at most* q *members.*

We see that we can apply this to routing on surfaces. If we have a large number of paths between 2 cuffs, then some number of them must be homotopic. This means we can treat the area they "live" as a disc. So we can basically think of working on things in the plane.

Question.

1. *Find a maximal* $k-$*system of cycles, pariwise non-homotpic on the torus, for* $k = 0, 1, 2$*.*
2. *What about if* k *is large?*
3. *What about* $k = 2$ *on the double taurus?*