Graph Minors

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1 Motivation

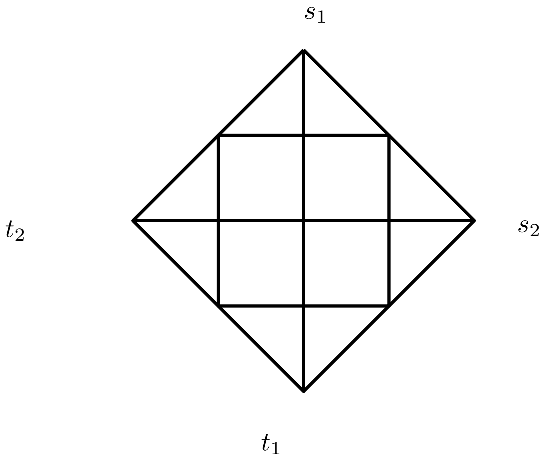
If we can obtain H from G via vertex deletions, we say $H \subset, G$. If we allow edge deletions we say $H \subset G$. If we allow edge contractions we say $H <_M G$ and say H is a minor of G. H is a proper minor of G if $H <_M G, H \neq G$.

The topic for the course is finding a structural decomposition for G in $Forb_H = \{G | H \not<_M G\}$.

5 good reasons to look at the structure theorem.

1. Polynomial time algorithm for $H-$minor testing if $H \leq_M G$
2. Proof of Wagner's Conjecture: In any infinite sequence of graphs $G_1, G_2, \ldots,$ $\exists\ i < j$ s.t. $G_i <_M G_j$
3. Poly Alg for testing membership in minor closed families. F is a minor closed class if $G \in F, H <_M G \Rightarrow H \in F$
4. Hadwiger's Conjecture: $K_t \not<_M G \Rightarrow \chi(G) \leq t - 1$
5. $k - DRP$: give G, $S = \{s_1, s_2, \ldots s_k\}, T = \{t_1, t_2, \ldots T_k\}, S, T \in V(G)$, do there exist disjoint paths $P_1, P_2, \ldots P_k$ such that P_i is from s_i to t_i.

Menger's theorem tells us that given 2 sets of size k then there exist k disjoint paths from S to T if and only if there is not a k vertex cut that separates S from T.



Model of H in G consists of a function im such that $\forall\, v \in V(H), im(v)$ is a tree of G. $\forall\, uv \in E(H), im(uv)$ is an edge of G s.t. $u \neq v \Rightarrow im(u) \cap im(v) = \phi$. $im(uv)$ has an endpoint in $im(u)$ and endpoint in $im(v)$.

The minor operation is transitive. If $F <_M G$ and $G <_M H$, then $F <_M H$. As are "is a subdivision" and "contains a subdivision"

Note that 1 and 2 together give us 3. Any minor closed family has a finite list of obstructions, since if the list was infinte then by 2 one would be a minor of another. Thus, since we have a poly time alg for each graph we get a poly time algorithm for the whole class.

We will see later than an algorithm for 5 also gives us a minor for 1.

G is a subdivision of H if $\exists$ distinct $c_v \in v(G)$ for each $v \in V(H)$. For each edge $uv \in H$, $\exists$ a path P_{uv} of G s.t. P_{uv} has endpoints c_u, c_v. P_{uv} and P_{wx} intersect only at common endpoints. If G contains a subdivision of H then $H <_M G$. The opposite is not true.

$\forall\, H \exists$ a finite set Z_H s.t. $H <_M G$ if and only if G contains a subdivision of some graph in Z_H. $Z_H = \{G$ s.t. there is a model of H in G s.t. no vertex of $im(v)$ which is not the endpoint if an edge image has degree 2$\}$

$Z_H = \{G | H <_M G$ but G does not contain t a subdivision of some $F \neq G$ s.t. $H <_M F\}$

Claim. *$H <_M G \Rightarrow \exists\, F \in Z_H$ s.t. G contains a subdivision of F*

Proof. By induction on $|V(G)|$. □

This shows how 5 gives us 1, but bruce erased the board he just finished writing, so I couldn't copy it down.

Let $H = K_{3,3}$ contract an edge. Then graphs without H have no K_5 or $K_{3,3}$, and so $H-$free graphs must be planar.

If $K_5 \not<_M$ and is edge with this property, then either G is planar, $G = W_8$, or G has a clique cut set of size ≤ 3. Consider a counter example to Hadwiger's Conjecture with $t = 5$ with $V(G)$ minimized and $E(G)$ maximized. Then it must be of one of the three types and we get a contradiction.

Suppose you are given an instance G, S, T of $k-$DRP and a clique of size $2k$ in G. Then either we can link through the clique and win, or we have a separating set of size at most $2k - 1$, so we can reduce the graph. Either way we can reduce to graphs which do not contain a K_{2k} minor.

Theorem. *$\forall$ k if (G, S, T) is an instance of $k-$DRP s.t. $S \cup T$ is attached to a K_{8k+5} model then the desired paths exist.*

What can we say if $K_t \not<_M G$? $K_3 \not<_M G \iff G$ is a forest. $K_5 \not<_M G$ includes planar graphs.

2 2-DRP

Theorem. *$2 - DRP$: If $(G, \{s_1, s_2\}, \{t_1, t_2\})$ is an instance of $2 - DRP$, $S \cup T$ is attached to a K_5 minor of G' iff the desired P_1, P_2 exist. We construct G' from G by adding edges between S and T, and a vertex x joined to $S \cup T$.*

Definition. Z is *attached* to a K_t model ($t \geq |Z|$) if for any set x with $|x| < 2$, the unique component of $G - X$ containing a vertex image intersects Z.

Proof. We prove this in 2 directions.

- $\Leftarrow$ Obvious

- $\Rightarrow$ Consider a minimum counterexample. If G' disconnected, consider just the component with the vertices. If there is a 1-separation, everything is on one side. If we have a 2-separation or 3-separation, we are again not minimal.

□

Claim. *G' has no 4-cut X s.t. some component U of $G - X$ is also disjoint from $S \cup T$ unless either $X = S \cup T$ or $\exists$ a unique U and it has only one vertex.*

Proof of claim. Suppose there exists such an X and U. □

Claim. *If we construct the graph $(X \cup U)'$, then the graph is planar.*

Proof. This graph has no $s'_1 - t'_1$ and $s'_2 - t'_2$ path and bruce erased the board too quickly again. □

Claim. *The graph obtained from $G' - U$ by replacing U with a 4-cycle on $(S \cup T)'$ plus a vertex u joined to all four vertices is planar.*

3 Planar Graphs

Lemma. *The faces of every 2-connected planar drawing are cycles.*

Lemma (Euler's Formula)**.** $|V| + |F| = |E| + 2$

Proof of both. By induction. Start with a cycle. We can either add an edge or add a vertex with 2 paths to the graph. Both results will hold. □

Theorem (Kuratowski's Thm)**.** *G is planar iff G does not contain K_5 and $K_{3,3}$ as a minor.*

Proof. We prove each direction.

$\Rightarrow$

$$\begin{array}{rcl} 2|E| & = & \sum_{f \in F} |bd(f)| \geq 3|F| \\ |F| & \leq & \frac{2}{3}|E| \\ |V| & \geq & \frac{1}{3}|E| + 2 \\ |E| & \leq & 3V - 6 \end{array}$$

For $G = K_5$, $|V| = 5, |E| = 10$.

For $K_{3,3}$, all faces have length at least 4, so we get $|E| \leq 2|V| - 4$ above and the result follows similarly.

$\Leftarrow$ Consider a minimal counterexample G that is not planar and contains no K_5 and $K_{3,3}$ minor. It is clear that G is at least 2-connected. Suppose we had a 2-cut. By adding an edge to the cut, we see that each side is planar and we are done by induction. Thus G is 3-connected.

Suppose we have a 3-cut. Then it must be free of edges, otherwise we can contract each side and get 2 planar minors. Consider a model of K_5 or $K_{3,3}$ in a component plus a triangle on the cutset. The model must use all edges in the triangle, otherwise the model would exist in G. Further, each edge of the triangle is an edge image, not a vertex image, since otherwise one would be redundant. Also, since the 3 vertices are pairwise adjacent it must be a model of K_5. But if we now contract the second component, we see that our original graph had a $K_{3,3}$. Thus, G is in fact 4-connected.

Consider G contract xy for some edge xy. This is planar by minimality. If we consider this embedding and delete the vertex formed by the edge xy. We have a cycle that is a face. We examine how the nerighbours of x and y behave on this cycle. There is a neighbour of x that is not a neighbour of y, otherwise we have a K_5 minor. Consider u which is a neighbour of x and not y. Then take the longest path in the cycle that contains no neighbour of y. This gives us a $K_{3,3}$ minor.

□

Theorem (Whitney's Theorem)**.** *3-connected graphs have unique planar embeddings.*

Proof. Suppose C is not a face boundary. Then it is a separating cycle. Suppose C is a face in some embedding. If there was a chord of the cycle, then we would have a 2-separation at its endpoints. Thus, every face is an induced cycle. Consider a component of $G - C$, then it is adjacent to every vertex of C, otherwise we have a 2-separation. Suppose we have 2 components, then each is adjacent to 3 vertices of C, add a vertex inside the face and we get a planar drawing of $K_{3,3}$. □

Theorem. *For an instance (G, S, T) of $k - DRP$ s.t. $S \cup T$ is attached to a near K_{4k} model, then the desired P_is exist.*

Definition. A *near clique model* for Z is a set of subgraphs $F_1 \dots F_t$ such that for each F_i, it is connected, or every component intersects Z, and for each pair of Fs that is disjoint from Z they are joined by an edge.

X spearates Z withg $Z \subset V, |Z| \leq t$ from a K_t model if $|X| < |Z|$ and the unique component of $G - X$ containing a vertex image is disjoint from Z. It is strongly attached if the only cut set of size $|X|$ is X.

Proof. If $S \cup T$ is strongly attached then we must have every vertex not in $S \cup T$ in a vertex image. Further, we must have each vertex image being singleton, and we are done.

Suppose we are not srongly attached. Suppose we have a cut of size $2k$, then we find paths from $S \cup T$ to the cut and we remain a near clique model. □

4 Routing in Planar Graphs

The $k - DRT$ problem is an instance $(G, X \subset V, |X| = k, \Delta_1, \dots, \Delta_p$ a partition of $X)$, do there exist vertex disjoint trees T_i such that $\Delta_i \in T_i$?

Definition. We say Δ is *realizable* in G if the desired trees exist. The trees are called a *realization*

$k-$realizations $(G, X), X \subset V(G), |X| = k$, which partitions of X are realizable in G?

There are 2 types of obstructions to not having the desired trees. Connectivity problems and topological problems.

Let $\text{crosses}_y = \{i | \Delta_i$ is not contained in a component of $G - Y\}$. If we have a Y such that $|\text{ crosses}_y| > |Y|$ then Δ is not realizeable. If we have some subset of Δ on a common face then if two of the $_i$s are interlaced there are no paths.

Lemma. *If X lies on the boundary of one face then Δ is realizeable iff one of the two above conditions fail.*

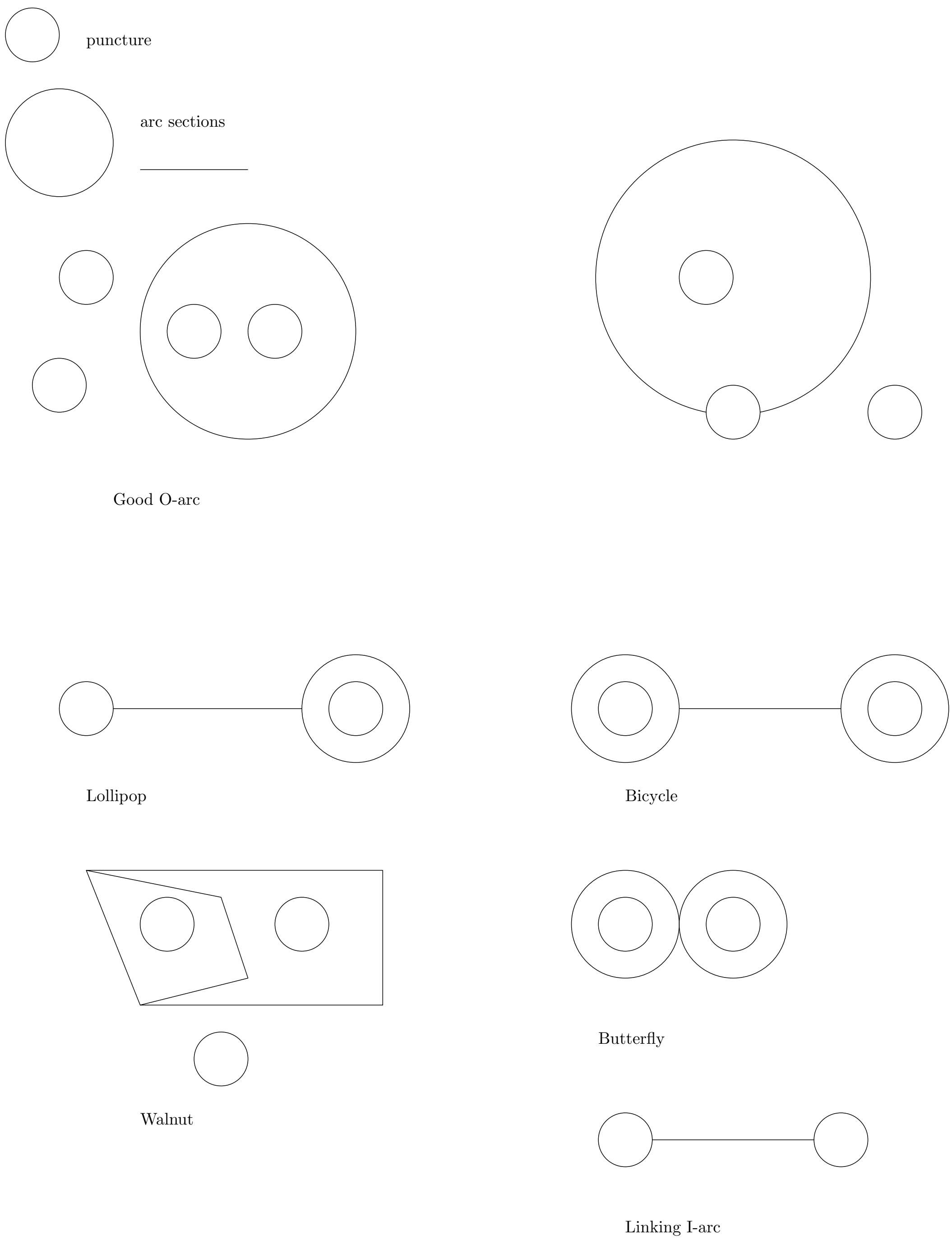
Definition. A *c-embedded k-Realization* problemn is an instance of k-realization such that G has a planar embedding with x contained in the boundary of c faces. Alternatively, we could say that G is embedded in a $c-$punctured plane Σ such that $G \cap bd(\Sigma) = X$.

Definition. Δ is *realizable* in Σ if $\exists$ disjoint trees of $\Sigma : T_1, \ldots T_p$ s.t. $\delta_i \subset T_i$.

Definition. A *cuff* is a component of the boundary of Σ.

Definition. An *I-arc* intersects G at only vertices and its endpouints are on cuffs.

Definition. An *O-arc* intersects G at only vertices and its interior is disjoint from cuffs.



Definition. A *schism* is any of a lollipop, bicycle, walnut, butterfly, good O-arc, linking I-arc, non-null-homotoipic looping I-arc

Theorem. $\forall\ c, k, \exists\ f(G,k)$ *s.t.* $f(G,x)$ *is an instance of* $G-$*embedd* $k-$*realizations s.t.*

1. *Every schism J satisfies $|J \cap V(G)| \geq f(x,k)$*

2. *for every O-arc J surrounding a cuff C, $|J \cap V(G)| \geq |X \cap C|$*

3. *for every non-null-homotopic looping I-arc J cutting off a cuff component D, $|int(J) \cap V(G)| \geq |D \cap X|$.*

then a partition of X is realizaeable in G iff Δ is realizeable in Σ.

Definition. For an instance (G,X) of c-embedded k-realizations V is *isolated* if there exist vertex disjoint cycles $C_1,\ldots,C_\ell$ of G bounding discs $D_1,\ldots,D_\ell$ s.t. $v \in D_1 \subset D_2 \ldots \subset D_\ell$ and D_ℓ intersects a cuff.

Corollary. *$\forall\ (c,k)\exists\ g(c,k)$ s.t. if v is $g(c,k)$ isolated, then Δ is realizable in G iff Δ is realizeable in $G-v$.*

Algorithm for solving $c-$embedded $k-$realizations. We induct on (c,k). Base cases $c=1, c=2$ done later.

Lemma. *In $O(V^2)$ time we can find either a $g(c,k)$ isolated vertex or a schism J with length $|J \cap V(G) \leq 8g(c,k)+5$. If we find a vertex, delete it. If we find a schism, we cut along it to get at most 3 instances of c', k' with $c' < c$ and $k' \leq k + 16g(c,k)+10$.*

Lemma. *For all v we can either find cycles $c_1,\ldots c_l$ showing that v is isolated or we can find a cycle of length $\leq 6\ell+2$ or an I-arc J_1 containing an O-arc J_2 surrounding a cuff s.t. $|J_1 \cup J_2 \cap V(G)| \leq 2\ell$*

Proof. Consider the face of $G-v$ containing v and show you get one of the listed things. Proceed by induction replacing v by some number of cycles which we can contract to a vertex v. □

5 Clique Pasting

Definition. A *Subtree Decomposition* for $G : [T, \{S_v | v \in V(G)\}]$, T is a tree, S_v is a subtree of T, $uv \in E(G) \Rightarrow S_u \cap S_v \neq \phi$.

This always exists, since we can consider the trivial 1 vertex tree.
$\forall\ t \in V(t), W(t) = \{v | t \in S_v\}$

Lemma. *G has a subtree decomposition s.t. W_t is a clique $\forall\ t$ if and only if $\S_u \cap S_v \neq phi$ G is chordal.*

***Insert Homework problem about common intersection.

Lemma. *If G arises from G_1, G_2 from clique pasting and G_1, G_2 have tree decompositions $[T_1, \{s_v^1, v \in V(G_1)\}]$, $[T_2, \{s_v^2, v \in V(G_2)\}]$. S.t. $\forall\ t \in T_1, G_1[W_t^1] \in F$ and $\forall\ t \in T_2, G_2[W_t^2] \in F$. Then G has a subtree decomposition s.t. $\forall\ t, G[W_t] \in F$.*

Since there is a vertex where the clique covers in each tree, we can add an edge between these two vertices and get a tree for G. And, in fact we can do this for an arbitrary number of pastings on the same clique.

Lemma. *G arises from the graphs in F via repeated clique pastings $\Leftrightarrow$ G has a subtree deconposition s.t. $\forall\ t \in V(t), G[W_t] \in F$ and $\forall\ st \in E(t), W_s \cap W_t$ is a clique.*

Definition. F^* is the set of graphs that arise from F via repeated clique pasting.

Lemma. *G is a spanning subgraph of a graph in F^* $\Leftrightarrow$ G has a subtree decomposition such that $\forall\ t$, $G^*[W_t]$ the graph obtained from $G[W_t]$ by ading edges so that $W_s \cap W_t$ is a clique forall $st \in E(T)$ is in F.*

Proof. We look at different clique sizes.

- If G has no K_1 minor then $G = \phi$.
- If $K_2 \not<_m G$ then we arise from clique pasting on a single vertex. We should have a tree decomposition where $|W_t| = 1 \forall\ t, F = \{\text{vertex}\}$.
- If $K_3 \not<_m G$ then we arise from clique pasting of vertices or edges.
- If $K_4 \not<_m G$ then if G is 3 connected, it's a clique of size at most 3.
- If $K_4 \not<_m G$ then
  - $\Rightarrow$ either G has a cutset X of size at most 2, or G is a clique of size ≤ 3.
  - $\Rightarrow$ either G is a subgraph if a graph which arises from smaller graphs with no K_4 minor via clique pasting or G has a clique of size ≤ 3.
  - $\Rightarrow$ G has a tree decomposition wher each $@_t$ has at most 3 vertices.
  - $\Rightarrow$ G ius a subgraph of a graph which has a tree decomposition where each W_t has $|W_t| \leq 3$ and $W_S \cap W_t$ is a clique $\forall\ st \in E(t)$.

□

Conjecture. *$\forall\, t$ s.t. $K_t \not<_m G \Rightarrow G$ has a tree decomposition s.t. $\forall\, W_s, |W_s| \leq f(t)$*

This conjecture is false.

1. Why is this false?
2. What is the right answer for $K_5 \not<_m G$?
3. What is the right answer for G has a subtree decomposition such that $\forall\, s, |W_s| \leq \ell$?.

Definition. A *bramble* is a set of trees every two of which intersect or are joined by an edge.

Definition. The *order* of a bramble is the minimum size of a set that hits every tree in it.

Examples:

- Clique model.
- Row $\cup$ column of a grid.

Lemma. *If G arises from $G_1, \ldots G_\ell$ via clique pasting then G has a bramble of order t if and only if $\exists\, i$ s.t. G_i has a bramble of order t.*

Proof. $t > |C|$ or else c contains a K_t which is a bramble oof order t in G and all G_i. Thus, C is not a hitting set for the bramble. So $\exists$ a bramble element B s.t. $B \cap C = \phi$, and $B \subset G_i - C$ for some i, $\forall$ bramble elements T, $V(T) \cap G$ induces a connected graph. □

G has a subtree decomposition such that $|W_s| \leq t \forall\, s \Rightarrow G$ has no bramble of order $l + 1$.

The conjecture is false because if we look at a $t+1 \times t+1$ grid, we have no K_5 minor, but the order of our bramble is $t+1$.

Theorem (Wagner's Theorem)**.** *$K_5 \not< MG \Rightarrow$*

- *G is planar*
- *G is W_8*
- *G has a cutset of size at most 2*
- *G has a cutset of size 3 whose deletion gives 3 components.*

Alternatively, we can state this as G is planar, G is W_8, or G is a subgraph of a graph that arises via clique pasting from smaller graphs with no K_5 minor.

This is equivalent to G having a tree decomposition where adding edges to make each $W_s \cap W_t$ a clique means each bag is planar or L.

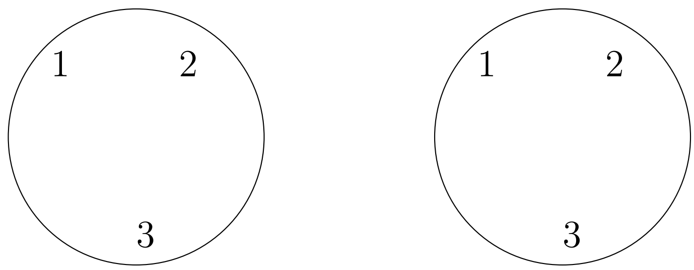
We now prove the base cases. Suppose $c = 1$. Recall this means that our graph can be embedded in the plane with all vertices on a common face. To check if a graph is embeddable this way, we add a vertex adjacent to the $S \cup T$ and see if this graph is embeddable. Call this graph G'. We may assume this is 2-connected. We will show that this is realizeable iff there is no s_1, s_2, t_1, t_2 appearing in that cyclic order around the face.

Lemma. *G' is $1-$realizeable iff there do not exist s_1, s_2, t_1, t_2 appearing in that cyclic order around the face.*

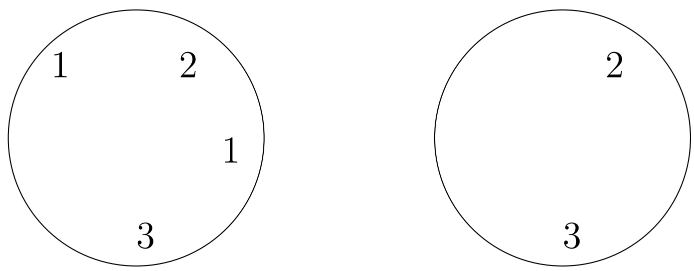
Proof. By an assignment problem, we either have such a crossing, a single partition element, or two consecutive partition elements. By induction the second 2 cases are realizeable. The first is not. This leads to an easy algorithm to check realizability in the graph. □

Lemma. *For $c = 2$, Δ is not realizeable in Σ iff we have either*

- *cross over one of the cuffs*
- *twisted triplet*

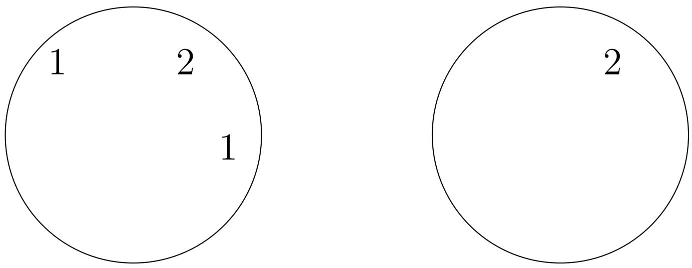


- *2 pairs crossing a face, cutting off a 3rd pair.*



Proof. Suppose we have none of the bad things, then we have one of the following things happening:

- There is a 1-element set.
- Δ has only 1 element
- Every Δ has its elements in one cuff
- Each Δ has one element in each cuff
- We have a pair on the same cuff with another pair split.

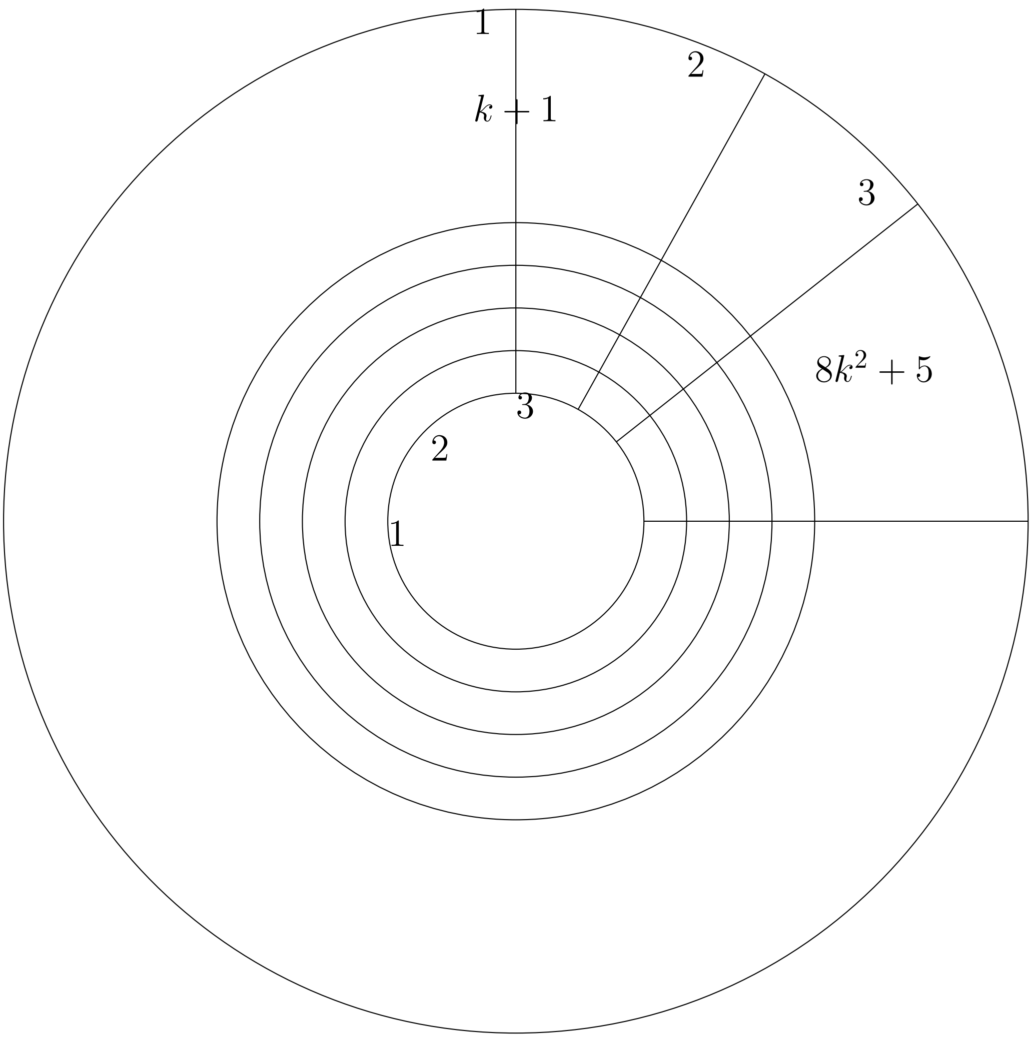


The first, second and fifth are easy to reduce to the graph from the surface.

The third we have to be cafreful because if we try to induce, we might have the graphs for the two separate cuffs intersecting. However, we note that if there are ℓ sets on one face, we need only vertices of distance at most ℓ to do our routing. So, we have two cases. Either there is a short arc between them, in which case we have a bicycle, cut along it, and can induct. Or, we have enough vertices between them to make the paths.

For the fourth, we again have some work. First,we try to find k disjoint paths between the cuffs. These exist iff there is no $O-$arc of size $< k$. If there is such an arc, paths are not poissible, so assume there is not one. We can apply Menger's theorem as an algorithm to find all minimal cuts of size $= k$. If for each consecutive pair of cuts there is a path of at most $8k^2 + 5$ vertices between them, we can cut along these and solve smaller problems. If no such path exists in some pair, then we show that between these 2 rings, all consecutive pairings are realizable.

It turns out that we must have a minor that is a large cylinder of internally disjoint paths which we can use to route along.



□

Theorem. *Second version of this this theorem. We will show this version implies the other version.*

$\forall$ *c*, *k*, $\exists$ *f*(*G*, *k*) *s.t.* *f*(*G*, *x*) *is an instance of G−embedd k− realizations s.t.*

1. *Every schism J satisfies* $|J \cap V(G)| \geq f(x,k)$
2. *If J surrounds a cuff that J has length* $\geq f(c,k)$.
3. *for every non-null-homotopic looping I-arc J cutting off a cuff component D,* $|int(J) \cap V(G)| \geq |D \cap X|$.

then a partition of X is realizaeable in G iff Δ *is realizeable in* Σ.

Proof - first version $\Rightarrow$ Corollary. Consider (G, X) and v satisfying the hypothesis. For $c = 1$, our previous proof shows $g(c, k) = k$ is sufficient. For $c = 2$, we recall we had seceral reductions we did in the previous case.

- solo vertex needs $g(c, k-1) + 1$
- Fated pair needs $g(c, k-1) + 1$
- If we treat the cuffs separately we need $g(c, k-1) + k$
- If we have a short linking $I-$arc we need $g(c-1, 5k) + 2k$
- Checking if X is in the same component $g(c, k) = 1$
- Each pair is split on the cuffs needs either $g(1, 16k^2 + k + 10) + 8k^2 + 5$ or $8k^2 + 7$ depending on whether we have a large ring, or all short rings.

For $c \geq 3$, let $h(c, k) = \max\limits_{c' < c, k' \leq k + 2f(c,k)} g(c', k')$. Then $g(c, k) = h(c, k) + f(c, k)$. Consider (G, X) and v satisfying the hypotheses. Assume for a contradiction $\exists$ a partition of X realizeable in G but not in $G - v$. This means one of our three conditions fails in $G - v$.

Suppose the first is violated. We can cut along this schism to get some new problem. There exists a realizaions of partitions of the elements in the boundary of the new problem whose union gives a realization of Δ in G. When we cut, v is still a distance of $h(c, k)$ from any cuff, and so by induction the graph is realizeable in the smaller case and was thus in the original case.

Suppose we have the second or third case. We again split into subproblems each of which is lexocographically smaller than the original, so we are done. □

Proof - version 2 $\Rightarrow$ version 1. We will actually show that if we get $f(c, k)$ for version 2 $\Rightarrow$ $3f(c, k)$ for version 1. Further, we will show a stronger version of the first theorem, where we insist:

- Lollipops have length $\geq 2f(c, k)$
- Good $O-$arcs have length $\geq 2f(c, k)$
- Bicycles have length $\geq 3f(c, k)$.
- Butterflies have length $\geq 2f(c, k)$
- Non-nullhomotopic looping $I-$arcs have length $\geq f(c, k)$
- Linking $I-$arcs have length $\geq f(c, k)$

- Walnuts have length $\geq 2f(c,k)$

Consider (G,X) satisfying the these new conditions but suppose for a contradiction that $\exists$ a partition of Δ which is realizable in Σ but not in G. Then the second condition of the second theorem does not hold. This means $\exists$ an $O-$arc surrounding a cuff C of length $\leq f(c,k)$. Choose one such that the complement of $\Sigma - J$ surroounding C is maximized. By Menger's threom, we have disjoint paths from the cuff to the $O-$arc.

If we look at the lengths of schisms in the new instance, we see this will satisfy the second theorem, and so by induction also satisfy the new theorem. $\square$

Proof - Second version. We proceeed by lexographic induction on (c,k). We show $f(x,k) \geq f(x',k')$ for $c' < c,\ k' \leq 5k$.

Let (G,X) be an instance of $c-$embedded $k-$realization satisfying the conditions of the theorem. Suppose for a constradiction that somne partition Δ if X is realizable in Σ but not in G.

Let J be a shortest linking $I-$arc. We claim there is an realization of Δ in $\Sigma\ J$ in at most k vertices of G. Furthermore, none of these intersection points is joined to an endpoint of J by a subarc of length $\leq 5k$.

$X' = X+$ both copies of these intersection points. We have a realization of a partition of Σ' of X' in G' which yields a partition of Δ. We need to show that (G',X') satisfies the conditions of the theorem. One place things could go wrong is linking $I-$arc, but any new one has length at least half as long as the one we cut, which is okay by induction. The other place is a looping $I-$arc where both ends are on the new cuff. If we have a cuff on each side of this, then we are done because the corresponding arcs in the old problem are long enough. If we do not, we must check the third condition and we see that this will still be satisfied unless the new arc links an existing cuff to the linking $I-$arc.

In this case, things could go wrong. So we have to change to cut along a bridge. A bridge is like a linking $I-arc$ except we hit each of the two cuffs twice. $\square$

New Section that's actually old.

Definition. The *width* of a subtree decomposition is $\max\{|W_t|-1, t \in V(T)\}$

Definition. The *Tree Width* of a graph G is the minimum width of a tree decomposition of G.

Theorem. $TW(G) = BN(G) - 1$

We will show this in 3 steps:

(A) $TW(G) \geq BN(G) - 1$

(B) $TW(G) \leq 3BN(G)$ *(Algorithmic)*

(C) $TW(G) \leq BN(G) - 1$

Let st be an edge of a subtree decomposition. Then if we delete this edge, we have components on each side, and components that intersect both vertices. We see that if there are components on each side then $W_s \cup W_t$ is a cut set in our original graph.

Similarly, if we delete a node of the tree, we see that any trees contained completely in one of the subtrees will be seperated from trees in other components when we delete W_s from the original graph.

For every bramble B of order $\geq 2k$, every minimal hitting set H for B is $k-$linked. (Ie. $\forall\ x \subset V, |X| \leq k, \exists$ a component U of $G - X$ with $H \cap U > \frac{H}{2}$

From this, we see that if a graph has $BN(2k)$ then the graph is $k-$linked, and if a graph is $k-$linked, we also have a bramble of order k.

Lemma. *If H_1, H_2 are hitting sets for B then $\exists$ $ord(B)$ vertex disjoint paths from H_1 to H_2.*

Proof. By menger's theorem this is only not true if there is a small cut set. But if there was such a cutset then the cutset would be a hitting set as well, but it's size is too small. □

Proof of Thm. We prove each piece separately.

(A) We claim that for every bramble B and subtree decomp. $[T, S]$, $\exists$ a node t s.t. W_t is a hitting set for B.

If this was true, we can do the following: Choose B with $ord(B) = BN(G)$ and $[T, S]$ of width $TW(G)$. So:

$$TW(G) = \text{width of } [T, S] + 1\} \geq |W_t| \geq ord(B) \geq BN(G)$$

We will prove the claim in 2 ways:

1. Supposed $\exists\ B, [S, T]$ for which the claim fails. This means $\forall\ st \in W(T)$, $W_s \cap W_t$ is not a hitting set for B means either $\exists$ a bramble element in V_s or $\exists$ a bramble element in V_t. Direct edges on tree towards hitting sets, then a sink will give a hitting set.

2. We need another claim. $\forall$ connected subgraphs C of G, $S_C = \{\bigcup_{v\in C} S_v\}$ is connected. This is obviously true.

   This tells us that $\exists\ t \in \bigcap_{c\in B} S_c$. And W_t is a hitting set for B.

(B) Let $(*)$ be he property G has no $k+1-$linked set. $(*) \Rightarrow \forall\ Z \subset V, |Z| \leq 2k+1, G$ has a tree decomposition of width $\leq 3k$ s.t. $\forall\ st \in E(T)$, $|W_s \cap W_t| \leq 2k+1$ and $Z \subset W_r$. $BN(G) = k \Rightarrow (*)$. We show that we can find a tree deconposition s.t. $\forall\ t, |W_t| \leq 3k+1$ and $\forall\ st, |W_s \cap W_t| \leq 2k$

Choose a root W_r and consider $G - W_r$ with components U_i and

Obs. 1 Every subgrph H of H satisfies $(*)$.

Obs. 2 We split the graph into the components of $G - W_r$.

Obs. 3 $W_r \cap W_{t_i}$ is vertices of W_r that have neighbours in U_i.

Obs. 4 $\forall\ Z \subset V, |Z| \leq 2k$ we ca get such a tree decomposition with $Z \subset W_r$.

Obs. 5 To prove the inductive statement it is enough to show that $\forall\ Z$ with $|Z| \leq 2k$ we can find $W supsetZ$ with $|W| \leq 3k+1$ s.t. for each component U_i of $G - W_r | x \in W | x$ has a neighbour in $U_i\}| \leq 2k+1$ and $\exists$ a vertex $vnot \in U_i$ and with no edgeto U_i.

We can assume G has at least $3k+2$ vertices and that $|Z| = 2k+1$. So Z is not $k+1-$linked, so $\exists\ X \subseteq V, |X| < k$ s.t. every component of $G - X$ contains $\leq k$ vertices of Z. $W = XcupZ$ works.

(C) We will prove that for any bramble B in G, exactly one of the following holds:

1. B extends to a bramble of order θ
2. $\exists$ a subtree dec $[T, S]$ s.t. for evert t either $|W_t| < \theta$ or W_t is not a hitting set for B and t is a leaf.

We proceed by backwards induction on the number of elements in B.

Case 1 $\exists$ a hitting set for B with $< \theta$ elements. Then 1. holds.

Case 2 is not case 1. Let H be a small hitting set for B. For any C_i s.t. C_i intersects or is joined by an edge to all $C \in B$ either 1 or 2 holds for $C_i \cup B$.

If it is 1, then it also holds for B and we are done. If it is 2 for all such C_i then for every C_i we can find a tree decomposition of $G[V(C_i) \cup H]$ s.t. $\forall\ t$ s.t. $|W_t| < \theta, W_t$ is a leaf and not a hitting set for B and $\exists$ a leaf t s.t. $W_t = H_t$.

If $\exists\ b \in B$ s.t. $C_i \cap b = \phi$, no $C_i \rightarrow b_{\text{edge}}$.

If $C_i \cup B$ is a bramble $\exists$ a subtree decomp. of G satisfying 2 for $C_i \cup B$, we want to massage it. So $\exists$ a leaf t s.t. W_t is a hitting set for B and does not intersect C_i. $\exists\ |H|$ v.d. paths $\{P_v | v \in H\}$ from H to W_t (in $G - C_i$). $\forall v \in H$ let Q_v be a path from S_v to t_i set $S'_v = S_v \cup Q_v$ for $v \in C_i$ set $S'_v = S_{v_i}$

□

We now show an algorithm which, given G of tree widh $\leq w$ and a set $Z \subset V$ with $|Z| \leq 2w + 2$, finds a subtree decomp. $[T, S]$ of G of width $\leq 3w + 3$. s.t. $Z \subseteq W_r$ for some node r.

(A) If $|V| \leq 3w + 3$ use a one ndoe tree.

(B) Otherwise

1. choose $Z' \supseteq Z$ with $|Z'| = 2w + 3$
2. Find $X \subset V, |X| \leq w+1$ s.t. $\forall$ components C of $G - X, |C \cap Z'| \leq w + 1$.
3. Set $W' = Z' \cup X$
4. For each component U of $G - W$, set $Z_u = \{x \in W | \exists\ y \in\ U, yx \in E(G)\}$
5. Recurse to find a tree decomposition $[T^u, \{S^u_v | v \in v(U) \cup Z^u\}]$
6. Paste these together

How many iterations do we carry out? How long does an iteration take? n iterations, but too long for each iteration.

We can do better. If we instead have $|Z| \leq 3w + 3$ and want to find a subree decomp. of width $\leq 4w + 4$. If we instead look for $U \cap Z' \leq 2w + 2$, then we can split the graph into 2 ppieces instead of components. It's like we're working on edges of the subtree instead of vertices.

This will run in $O(n^2)$ time

We could go one step further and double out constants. This will allow us to split so that both vertices and edges are at most 2/3 in the components. This will run in $O(n \log(n))$ time.

Ideally, we would like to fund a tree decomposition $[T, S]$ such that

- for every maximal bramble B, $\exists$ a unique t_B such that W_{t_B} is a hitting set for B
- $\forall\ B_1, B_2, \exists$ an arc st of the path of T_{B_1} to t_{B_2} s.t. $W_s \cap W_t$ separates a hitting set for B from a hitting set for B_2 and is a smallest such set.
- $t_{B_1} = t_{B_2}$ iff one is "contained" in the other.

Brambles as Biases

$\forall$ brambles B of order k we have a bias $f_B : \{x \subseteq V|\ |X| < k\}$, $f_B(X) =$ the unique component of $G - X$ containing an element of B.

At this point, I stopped taking good notes so that I could better pay attention and understand the materiaol myself.

Definition. A *tangle* is a bramble where any 3 elements have an edge which has each at one endpoint.

A bramble of order $3k$ gives a tangle of order k, so we don't lose much.

Tree decompositions are equivalent to a set of laminar separations.

For a tangle T, a set Z is *attached* to T if $\exists\ Y \subseteq V, |Y| < |Z|$ s.t. (Y separatews Z from a hitting set for the tangle $\Leftrightarrow$ Z is disjoint from $f_T(Y)$).

We say that a $k \times k$ grid is attached to T if the bramble crosses$_k$ is indistinguishable from T, $ord(T) \geq K$.

Our goal is to show:

Theorem. *$\forall\ k, \exists\ f(k)$ s.t. for any tangle T of order $f(k)$ there is a $k \times k$ grid minor attached to T.*

Theorem. *If G is a planar graph with a tangle of order $4k+1$ then is has a $k \times k$ grid as a minor.*

Proof. For any $O-$arc of length $< 4k+1$, the tangle is inside J or outside J. Take the smallest $O-$arc where the tangle is inside. $|V(G) \cap J| = 4k$, and we consider the vertices of this arc and see we have rk vertices around a face, so by the assignment question, we have a $k \times k$ grid. □

Theorem. *If we have a tangle of order 20^{2k^5} then we have a $k \times k$ grid minor attached to T. We will only show $20^{k^{4k+1}}$, since it's easier.*

Z is $T-$linked if $|Z| \leq ord(T)$ and $\forall\ Y$ with $|Y| < |Z|$, Z intersects $F_T(Y)$, equivalently does not separate Z from a hitting set for T.

A model of the $k \times k$ grid is attached to T if $ord(T) \geq k$ and $\forall\ Y$ with $|Y| < k$,$F_T(Y)$ is the unique component of $G - Y$ containing the image of a row and column.