# Induced Subgraphs

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**Definition.** $H$ is an *induced subgraph* of $G$ if $H$ is a subgraph of $G$ and if $u, v \in V(H)$ and $u\ v$ in $G$ then $u\ v$ in $H$.

**Definition.** A graph is *chordal* if it contains no hole.

**Definition.** A *hole* is an induced cyle of length at least 4.

**Definition.** An *interval graph* is the intersection graph of a family of intervals.

**Definition.** An interval graph is *proper* if no interval is a subset of another.

**Definition.** The *Line Graph* of $G$, $L(G)$ is the intersection graph of the edges of $G$.

**Definition.** A graph is *perfect* if $\chi(G) = w(G)$ for every induced subgraph.

How do we recognize perfect graphs? How do we construct all perfect graphs? Which minimal graphs are not perfect?

**Theorem.** *Strong Perfect Graph Conjecture: the obstructions for being perfect are odd holes and odd antiholes.*

**Theorem.** *Weak Perfect Graph Conjecture: the complement of a perfect graph is perfect.*

Graphs that are perfect include:

- Bipartite graphs
- Complements of bipartite graphs

- Line graphs of bipartite graphs
- Complement of line graphs of bipartite graphs
- Chordal Graphs
- Split Graphs
- Comparability Graphs
- Complements of Comparability Graphs

**Definition.** A *1-join* of $G$ and $H$ is when you take a vertex of $G$ and a vertex of $H$ and join their neighbourhoods by a complete bipartite graph. A *2-join* is similar but doing it twice to the same pair.

**Definition.** A *Berge Graph* is a graph with no odd hole and no odd antihole.

**Definition.** An *even pair* of vertices in a graph is a pair of vertices $u, v$ such that every induced $uv$ path has even (edge) length.

**Definition.** A *claw* is $K_{1,3}$. A graph is *clawfree* if it has no induced claw.

How can we construct all clawfree graphs? What graphs are clawfree?

- Linegraphs
- Icosahedron
- Schlafli graph
- Circular interval graphs
- Complements of triangle free graphs

**Definition.** The *schlafli graph* is taken by taking 3 disjoint copies of the line graph of $K_{3,3}$ and joining special 3 sets by special 6 cycles

**Definition.** A *circular interval graph* is a cycle where some intervals are replaced by cliques.

**Definition.** A *pyramid* is a subdivision of $K_4$ where one triangle is just a triangle and at least 2 of the others are paths.

**Lemma.** *If $G$ is chordal and $X$ is a minimal cutset, then $X$ is a clique.*

*Proof.* Suppose there exists a cut that is not a clique. Each vertex of the cut has a neighbour in each component of $G-X$. Consider 2 vertices $u, v$ of the cut that are not adjacent. Choose 2 components of $G-X$. A shortest $uv$ path in each component is induced. The union of two of these paths from different components gives us a hole. This contradicts $G$ being chordal. □

**Definition.** A vertex $v$ is *simplicial* if $nbhd(v)$ forms a clique.

**Lemma.** *Every non-empty chordal graph has a simplicial vertex*

**Lemma.** *If $G$ is chordal and $C$ is a clique, then there is a simplicial vertex not in $C$.*

*Proof.* By induction. □

Let $T$ be a tree and $T_1$ to $T_n$ be subtrees of $T$. Construct a graph with $v_1, v_2, \ldots v_n$ with $v_i\ v_j$ if $V(T_i \cap T_j) \neq \phi$ and $i \neq j$. $G$ is the intersection graph of the subtrees of trees.

**Lemma.** *$G$ is chordal iff it is a subtree intersection graph.*

*Proof.* Suppose Intersection tree graph has a hole. We have $T_1 \sim T_2, T_2 \sim T_3, T_3 \sim T_4, T_4 \sim T_1$. Delete $T_2 \cap (T_1 \cup T_3)$. Then $T_1$ and $T_4$ are now disconnected. So $T_3$ must have a vertex that intersects $T_1$.

For the other direction, we prove by induction. Let $v$ be a simplicial vertex of $G$. By induction, $G-v$ is the intersection graph of subtrees $T_1, \ldots T_n$ of $T$. The neighbours of $v$ are a clique and so they pairwise meet, and thus must all intersect in a common vertex. Create a vertex adjacent to that vertex and add it to all the proper trees and make $T_v$ be that vertex. □

include image LineGraphObstructions

**Theorem.** *$G$ is the line graph of a triangle free graph iff no induced subgraph of $G$ is the claw or the diamond.*

*Proof.* One direction is obvious. For the other, let $A_1, \ldots A_n$ be the maximal cliques. If any two of these intersected in more than one place, we would have a diamond as an induced subgraph, which is a contradiction. Further, if any vertex were in more than 2 maximal cliques, we would have a claw. So let $H$ be the intersection graph of the maximal cliques of $G$. Then any vertex of $G$ corresponds to 1 or 2 vertrices of $H$. For any that corresponds to a single vertex add a leaf vertex to the graph there. This has no triangles, so we are done. □

**Theorem.** *For bipartite line graphs, we add odd holes.*

**Theorem** (Lekkerkerker and Boland)**.** *Minimal non-interval graphs are holes, asteroidal triples*

**Definition.** An asteroidal triple is a stable set of 3 vertices such that there exists a path between any pair that is not adjacent to the third.

**Theorem.** *Or, we have*

- *a claw with talons of length 2*
- *a path of length 5 plus a center, plus a leaf on the middle*
- *A path of length* $\geq 2$ *plus a center, plus leaves on center and end of path*
- *A path of length* $\geq 2$ *plus 2 centers, plus a vertex to the 2 centers, plus 2 vertices each to one center and one path end*

**Definition.** A *circular arc graph* is the intersection graph of arcs of a circle.

What graphs are minimal not circular arc graphs? Unknown.

**Definition.** A *comperability graph* is a graph defined on a poset, where $u, v$ are adjacent if $u \geq v$ or $v \geq u$.

There are approximately 19 types of induced subgraph obstructions for this.

**Definition.** A *linear Interval Graph* is a set of $k$ vertices and we choose subsets of consecutive vertices and put a clique on it.

**Theorem.** *$G$ is a Proper Interval Graph iff $G$ is a Linear Interval Graph*

**Definition.** A *Circular Interval Graph* is vertices on a circle

**Theorem.** *$G$ is a proper circular arc graph iff $G$ is a circular interval graph.*

**Lemma** (Gallai's identities)**.** *For any graph $G$:*

- *min $k$, $\exists$ set of $k$ vertices meeting all edges + mak $k$ $\exists$ stable set of size* $k = |V(G)|$
- *mink $k$, $\exists$ $k$ edges covering all vertices + max* $k, \exists$ *matching of size* $k = |V(G)|$.

*Proof - Weak Perfect Graph Theorem.* We show that a graph $H$ is perferct iff $\alpha(H)\omega(H) \geq |V(H)|$. The only if direction is obvious, so we show the if direction. By induction, WMA every proper induced subgraph is perfect.

Suppose there exists a stable set that intersects all max cliques, then we win by induction. So we assume there is no such stable set. Consider a max stable se. If we delete a vertex of the stable set, the remaining graph is colourable, so we get some stable sets. Do this over all vertices in the stable set. And take the collection of all these stable sets. Each vertex occurs in $\alpha(G)$ of these sets, and for each such set there is a maximal clique that vertex does not meet. □

We woud like to show that every Berge graph can be built starting from known classes of perfect graphs combined by some set of operations that maintain being perfect. As basic graphs, we need:

- bipartite graphs
- complement of bipartite graphs
- line graphs of bipartite graphs
- complements of line graphs of biparite graphs
- double split graphs.

**Definition.** A *double split graph* is constructed by taking a clique and a stable set with some edges between them. We twin each vertex, and join the twins in the independent set and not join them in the clique. We then add edges across by a matching with parity 0 if there was an edge, parity 1 if there was no edge.

**Definition.** A *1-join* is constructed by taking $G, H$, deleting a vertex from each and forming a complete bipartite graph on their neighbourhood.

**Claim.** *If $G$, $H$ are perfect, then a 1 join is also perfect and is Berge.*

*Proof.* We examine how any odd hole or antihole must interact with the cutset and see it is Berge. To see it is perfect, □

# Maria

**Conjecture** (Erdos Hajnal Conjecture)**.** *For every $H$, there exists $\delta(H) > 0$ such that if $G$ does not have an induced subgraph isomorphic to $H$, then either $\omega(G) \geq |V(G)|^{\delta}(H)$ or $\alpha(G) \geq |V(G)|^{\delta}(H)$.*

A graph has the EH property if there exists such a delta.

- for an edge, $\delta = 1$
- for $K_t$, $\delta = \frac{1}{t-1}$
- 3-edge graphs give the 2 edge path. These graphs are disjoin unions of cliques, so $\delta = \sqrt{n}$.

**Definition.** Given graphs $H_1, H_2$ and $v \in H_1$, then $F$ is obtained by *substituing* $H_2$ in $H_1$ for $v$ by deleting $v$ and joining $H_2$ to the neighbours of $v$.

**Theorem.** *If $H_1, H_2$ have EH property with $\delta_1, \delta_2$, then $F$ has EH with $\delta = \frac{\delta_1\delta_2}{\delta_1+n_1\delta_2}$.*

*Proof.* Let $M = n^{\frac{\delta_2}{\delta_1+n_1\delta_2}}$. Every $M-$subset of $V(G)$ contains $H_1$. So $G$ must have at least $\frac{\binom{n}{m}}{\binom{n-n_+1}{M-n_1}}$ copeis of $H_1$, and at most $\binom{n}{n_1-1}$ copies of $H_1 - v$. Some copy of $H_1 - v$ extends to a copy of $H_1$ in at least $\frac{\binom{n}{m}}{\binom{n-n_1}{M-n_1}\binom{n}{n_1-1}} \geq n^{\frac{\delta_1}{\delta_1+\delta_2 n_1}}$.

Let $W$ be the set of vertices extending this copy of $H_1 - v$ to $H_1$. By the size of this set, it either has a big clique or big stable set, or $H_2$. □

**Theorem** (Erdos Hajnal)**.** *$\forall H$, if $H \not\subset_i G$, then either $\alpha(G) \geq e^{c(H)\sqrt{\log(n)}}$*

**Theorem** (Erdos, Hajnal, Pach)**.** *$H \not\subset_i$ then we have 2 sets of size $\delta(H)$ such that they are complete or anticomplete to each other.*

**Theorem** (Fox, Sudakov)**.** *We get either a clique or a pair of anticomplete sets.*

**Definition.** A set $X$ is homogeneous if all vertices have the same set of neighbours.

**Definition.** A graph $G$ is *$\alpha$-narrow* if for every function $g$ s.t. $\forall$ perfect induced subgraphs $P$ of $G$, $\sum_{v\in P} g(v) \leq 1$ we have $\sum_{v\in G} g^\alpha(v) \leq 1$.

**Theorem** (Chudnovsky, Safra)**.** *$\delta($The Bull$) = 1/4$.*

*Proof.* To see this is best possible consider a triangle free graph with a stable set of size $\sqrt{n}$. Substitute the compliment of this for all vertices of itself. It will have $n^2$ vertices, be bull free and have $\alpha, \omega = 1/4$.

If $G$ is perfect then $\delta(G)\omega(G) \geq |V(G)|$. So if $G$ has a perfect induced subgraph of size $n^\epsilon$ then it has $\omega$ or $\alpha$ at least $n^{\epsilon/2}$.

So it suffices to show that every bull free graph has a perfect graph with at least $n^{\frac{1}{2}}$ vertices. □

**Theorem.** *If $G$ is $\alpha$-narrow then $G$ has a perfect induced subgraph of size $\geq n^{\frac{1}{\alpha}}$.*

*Proof.* Let $P$ ne a perfect induced subgraph of $G$ with $|V(P)|$ is maximized and equals $k$. Let $g(v) = \frac{1}{k}$. Now since $G$ is $\alpha-$narrow, so $k \geq n^{\frac{1}{\alpha}}$. □

**Theorem.** *Every bullfree graph is $2-$narrow.*

**Definition.** A bullfree graph is *basic* if it does not contain a hole of size $\geq 5$ with a center and an anticenter

**Theorem.** *If $G$ is bullfree, then either $G$ is basic or $G$ has a homogenous set .*

*Proof.* We proceed by induction and consider 2 things.

1. Every basic graph is 2-narrow

   We want to show that $G$ is basic $\Rightarrow \forall v$ either $N = G|N(v)$ is perect or $M = G - (N(v) \cup v)$ is perfect. It is enough to show that either $N$ or $M$ is Berge. Consider the case where $M$ has an antihole and $N$ has a hole.

   Any vertex in the antihole has at most 2 non neighbours in the hole and any vertex in the hole has at most 2 neighbours in the antihole, otherwise we will have a bull or a center. The total number of edge locations between the cycles is $ab$ and we must have at least $a(a-2) + b(b-2)$ total possible adjacencies. For this to be true, one of $a$ or $b$ must be at most 4.

   Assume we have a $G$ that is basic and minimal not $2-$narrow. Let $v_0 \in V(G)$ be s.t. $g(v_0)$ is maximized. By above and taking compliments, we can assume that $N(v_0) \cup v_0$ is perfect. Define $g' : M \to [0, 1]$, $G'(v) = \frac{g(v)}{1-g(v_0)}$. Since $g$ is good, deduce that $g'$ is good on $G|M$. By the minimality of $G$, $G|M$ is $2-$narrow. So we have $\sum g^2(v) \leq \sum(1 - g(v_0))^2$.

   And $\sum g^2(v) = g(v_0)[g(v_0) + \sum\limits_{v \in N} g(v)] + (1 - g(v_0))^2 \leq 1$.

2. $G$ admits a homogeneous set decomposition.

   We can unsubstitute the homogeneous set to get 2 seperate graphs. Let $g$ be a good function on $G$. We need to prove that $\sum g^2(v) \leq 1$. Let $g_1$ on $G_1$ (the homogenous set) $= g(v)/(\max_{\text{perf} \in G_1} \sum g(v)$. This is a good function. $g_2(v) = g(v)$ except for the vertex which is $G_1$, which gets weither $k = \max$ from previous case. This is also a good function. We can combine the functions to get the sum of the squares of the whole graph is at most 1.

□

**Theorem** (Hayward's Theorem)**.** *If $G$ has no $P_4$ and no $\bar{P}_4$, then either $G$ is perfect or $G$ is $C_5$, or $G$ admits a homogeneous set decomposition.*

**Corollary.** *Graphs with no $P_4$ or $\bar{P}_4$ are $2-$narrow.*

*Proof of Thm.* Assume $G$ is not perfet. Then $G$ contains a 5-gon. All other vertices are centers, anitcenters, or clones of vertices on the 5-gon. If there is a clone, then the clones of that vertex will form a homogeneous set decomposition.

So now we have all vertices are either centers or anticenters to the 5-gon and the $5-$gon is itself a homogeneous set. □

**Definition.** A class $C$ of graphs is *$\chi-$bounded* if $\forall\ G \in C, \forall\ H \subseteq G$, $\exists\ f\colon N \to N$ such that $\chi(H) \leq f(\omega(H))$

In general $H$-free graphs are not $\chi$ bounded, since we can have graphs of arbitrarlity large girth with large chromatic number. If there is a cycle in $H$, we are screwed.

**Conjecture.** *For every tree $T$, the class of $T-$free graphis is $\chi-$bounded.*

**Conjecture.** *$Forb(C_5, C_7, \ldots)$ is $\chi-$bounded*

**Conjecture.** *For every $t$, $Forb(C_t, C_{t+1}, \ldots)$ is $\chi-$bounded.*

**Lemma.** *Conjecture 1 is true for paths with $\chi(G) \leq (t/2)^\omega$*

**Definition.** $Forb*(F)$ is the class of graphs with no subgraph isomorphic to a subdivision of a member of $F$.

**Theorem.** *For every tree $T$, $Forb*(T)$ is $\chi-$bounded.*

**Theorem.** *$Forb*(bull)$ is $\chi-$bounded.*

**Conjecture.** *$Forb*(H)$ is $\chi-$bounded.*

**Theorem.** *For every $t$, $Forb(C_5, C_7, \ldots, C_t, C_{t+1}, C_{t+2}, \ldots)$*