1 Posets and Extremal Set Theory

Definition. A partially ordered set, or a poset, consists of a set P (assumed to be finite, but not required) and a relation $\leq$ on P which satisfies:

$$\text{reflexive } \forall x \in P x \quad \leq x$$

$$\text{antisymmetric } \forall x, y \in P \quad (x \leq y \text{ and } y \leq x) \Rightarrow x = y$$

$$\text{transitive } \forall x, y, z \in P \quad (x \leq y \text{ and } y \leq z) \Rightarrow x \leq z$$

Example

- $P = \{a, b, c, d\}$
  $\{a \leq a.b \leq b, c \leq c, d \leq d, a \leq b, b \leq c, a \leq c, d \leq c\}$
  ***Combinatorics 1
- $S = \{1, 2, 3\} P = 2^S \; X \leq Y \iff X \subset Y$
  ***Combinatorics 2
- $S = \{1, 2, \ldots, n\} P = 2^S \; X \leq Y \iff X \subset Y$

Definition. If $x \leq y$ and $x \neq y$ then we write $x < y$

Definition. An element $x \in P$ is maximal if there does not exist $y \in P$ such that $y > x$

Definition. An element $x \in P$ is minimal if there does not exist $y \in P$ such that $x > y$

Definition. x and y are comparable if either $x \leq y$ or $y \leq x$. Otherwise they are incomparable.

Definition. A chain is a subset $X \subset P$ such that for any pair $x, y \in P$, x, y are comparable.

Definition. A chain is a subset $X \subset P$ such that for any pair $x, y \in P$, x, y are incomparable.

Theorem. *If P is a poset with largest chain of size k then P can be partitioned into k antichains.*

Proof. We proceed by induction on k. Consider the set X of all maximal elements. This is a an anti-chain. By induction $P - X$ can be partitioned into $k - 1$ antichains, so we're done. □

Theorem. *If P is a poset with largest antichain of size k then P can be partitioned into k chains.*

Proof. We proceed by induction on $|P|$. Let X be a maximal chain in P. If $X = P$ we're done. If the largest antichain in $P - X$ has size $\leq k-1$ then we win by induction. So WMA $P - X$ has an antichain Y of size k.

We want to partition X into X^+, X^- st. $\forall x \in X^+, \forall y \in Y, x \not< y$, $\forall x \in X^- \forall y \in Y x \not> y$.

Set $X^+ = \{x \in X : x > y \text{ for some } y \in Y\}, X^- = X - X^+$.

For every $x \in X$ either $x \not> y$ for every $y \in Y$ or $x \not< y$ for every $y \in Y$.

We can partition the poset into things larger than Y and things smaller than Y. We can apply induction to these two sets and win. □

Definition. For a poset P, the comparability digraph associated with P has vertex set P and an edge (x, y) if $x > y$.

Definition. The height of a poset is the length of the longest chain.

Definition. the width of a poset is the size of the longest antichain.

Definition. If D is an acyclic digraph, the transitive closure of D is the digraph formed by joining by an edge any two vertices that are joined by a directed path.

Theorem. *(Gallai-Roy-Vitauer) If D is a loopless digraph with longest directed path of length k then D is $(k+1)$ colourable.*

Theorem. *(Gallai-Milgram) If D is a digraph then it has a collection of $\alpha(D)$ directed paths with partition the vertex set.*

Let $L_1, L_2, \ldots, L_m$ be linear orders of Ω. Define a relation $\leq$ on Ω by the rule $x \leq y$ if $x \leq_{L_i} y$ for every $1 \leq i \leq m$. It is easy to see that $\leq$ is a partial order.

If L is a linear order on Ω, P is a partial order on Ω then $L is a linear extension of \text{P} is P \subset L$.

Observation: Every partial order is an intersection of linear extensions.

Proof. Suppose x, y are incomparable then there is a linear extension L_{xy} with $x \geq y$. Take the intersection of L_{xy}, L_{yx} for all incomparable x, y. □

The dimension of a poset P is the minimum number of linear extenstions with intersection equal to P.

Conjecture. *For every non-linear poset P there exist incomparable x, y st $x \geq_L y$ in $\geq \frac{1}{3}$ of all linear extensions L.*

Theorem. *For every poset P, $dim(P) \leq width(p)$.*

Lemma. *If C is a chain in P then there exists a linear extension L of P such that $\forall x \in C, \forall y \notin C, x \geq_L y$ if x, y are incomparable in P.*

Proof. Let $C = \{x_1, \dots x_n\} x_1 \geq x_2 \cdots \geq x_n$. For $i = 2, \dots n$, let $A_i = \{y \in \Omega / C | y \geq x_i, y \ngeq x_{i-1}$. $B = \Omega/(C \cup A_1 \cup \cdots a_n)$ □

Proof. (of Thm) Set $k = \text{width}(P)$. Partition Ω into k chains $C_i, \dots, C_k$ and apply the Lemma to choose a linear extension L_i of P for each chain C_i. Then $\cup_{L_i} = P$

□

Theorem. *Let $G = (V, E)$ be a simple graph, P the associated poset. Then dimension of $P \leq 3 \iff G$ is planar.*

Proof. ($\Rightarrow$) Let L_1, L_2, L_3 be linear extensions with $L_1 \cup L_2 \cup L_3 = P$. □

Let $\Omega = \{a, b, c, \dots, z\}$ For $i = 1, 2, 3$, choose $a_i, b_i, c_i, \dots z_i \in R$ st, $x_i \leq y_i \iff x \leq_{L_i} y$

$x \to (x_1, x_2, x_3)$ is an embedding of Ω in R^3. Proeject onto the hyperplane $H = \{(\alpha_1, \alpha_2, \alpha_3) : \alpha_1 + \alpha_2 + \alpha_3 = 0\}$ WMA Ω is embedded in H.

Show that the edges don't cross (non-trivial)

($\Leftarrow$) We may assume we have a triangulation. We can assume it has a magic colouring, where each triangle has colours 1,2,3 in clockwise order on the angles and each vertex has 1,2,3 appearing in clockwise order, with repetition.

Definition. A vertex is type α for a cycle C if C seperates the labels α from vertex α.

Claim: Every cycle has a vertex of type 1,2,3.

Proof. Suppose no type 1 vertex. Suppose a chord, contradiction. Suppose an interior vertex, contradiction. Then C is a triangulation.

Construct D_i if there is a triangle containing x, y with 1 at the x angle, then $x \geq y$. Extend to a linear order since this is acyclic (by claim). Extend D_i to linear order on V. Extend to linear order on $V \cup E$ by placing each edge above its largest endpoint. □

Theorem. *(Sperner's Theorem) If $A_1, A_2, \dots A_m \in \{1, 2, \dots, n\}$ and $a_i \not\subset A_j$ then $m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.*

Proof. Let $a_i = |A_i|$. Let C_i be the set of all maximal chains in the poset of subsets containing A_i. $C_i \cap C_j = \phi$ and $|C_i| = a!(n-a)! \geq \lfloor \frac{n}{2} \rfloor! \lceil \frac{n}{2} \rceil!$. So $n! \geq \sum |C_i| \geq m \geq \lfloor \frac{n}{2} \rfloor! \lceil \frac{n}{2} \rceil!$ □

Theorem. *If $A_1, A_2, \ldots A_m \in \{1, 2, \ldots, n\}$ and $A_i \cap A_j \neq \phi$ then $m \leq \binom{n-1}{k-1}$.*

Proof. Let $F_i = \{i, i+1, \ldots, i+k-1\}$ At most k of the F_is can be A_is. For any permutation $\pi \in S_n$, let $F_i^\pi = \pi(F_i)$. Any k element subset appears as an F_i in exactly $nk!(n-k)!$ different permutations π.

Then $mnk!(n-k)! \leq kn!$. Thus m $\leq \binom{n-1}{k-1}$. □

Definition. The Kneser graph $Kn(n, k)$ has vertex set of $k-$subsets of $\{1, 2, \ldots, n\}$ and two vertices are adjacent if the corresponding sets are disjoint.

Indpendence number of $Kn(n, k) = \binom{n-1}{k-1}$

2 Design Theory

Definition. An incidence structure is a triple $(\mathcal{P}, \mathcal{B}, \mathcal{I})$ where P are points, B are blocks, $I \subset P \cdot B$.

v is the number of points, b is the number of blocks, k size of the blocks, every t element subset is in λ blocks.

Definition. An incidence structure is simple if no two blocks are incident with the same points.

For convenience, we treat blocks as subsets of points and say that a point $x \in B$ a block is $x, BinI$.

Definition. An incidence structure is a linear space if $|B| \geq 2$ for every $B \in \mathcal{B}$ and $\forall x, y \in \sqrt{}, !\exists B, x, y \in B$.

Theorem. *If S is a linear space with at least 2 blocks then $b \geq v$.*

Proof. Let r_x be the number of blocks containing x. Let k_B be the number of points in B. We may assume $b \leq v$. Then we have $v(b - r_x) \leq b(v - k_b)$ □

$$\sum_{x \in P} \sum_{b \in \mathcal{B}, x \notin B} \frac{1}{v(b-r_x)} \geq \sum \sum \frac{1}{b(v-k_B)}$$

Definition. A $t-(v, k, \lambda)$ design is an incidence structure on v vertices with the following properties:

- Every block has size k.
- Every subset of t points is contained in exactly λ blocks.

This is also called a $S_\lambda(t, k, v)$

Proposition. *In a $t-(v, k, \lambda)$ design, $b\binom{k}{t} = \lambda\binom{v}{t}$*

Proof. Each t points must occur in λ blocks. There are $\binom{b}{t}$ such sets. Each block contains $\binom{k}{t}$ such $t-$sets. □

Definition. 2-designs are called Balanced Incomplete Block Designs (BIBDs)

Let $P = E(K_5)$. A subset $B \in P$ is a block if it is one of $K_{1,4}, K_3 \cup K_2, C_4$. There are 10 edges, and each block has size 4. The possible three edge subsets are $K_3, P_3, K_{1,3}, K_2 \cup P_2$. Each occurs in exactly one block, so we have a $S_1(10, 4, 3)$ design.

Let $P = Z_2^4$. $B = \{\{w, x, y, w\} \subset Z_2^4 : w + x + y + z = 0\}$. There are 16 elements, each block has size 4, and for any three element subset, there is a unique fourth element that forms a set with it. So we have a $S(3, 4, 16)$ design. In fact, we can get an $S(a, a+1, (a+1)^2)$ design by this construction.

The Fano plane is an $S(2, 3, 7)$ design. ***Combinatorics3

We can construct a bipartite graph with points on one side and blocks on the other. This is called the Levi graph.

An incidence structure $S = (P, L)$ is a projective plane if it satisfies:

- Every two distinct points lie on exactly one line
- Every two distinct lines intersect in exactly one point
- There are 4 points, not three of which lie on a line
- Every line has the same size

Theorem. *If $S = (P, L)$ is a projective plane, then there exists a number n called the order of S, such that*

- *Every point is inc with $n + 1$ lines*
- *Every line is inc with $n + 1$ points*
- $v = n^2 + n + 1$
- $b = n^2 + n + 1$

Proof. Consider a point x and a line l not on that point. The number of lines through x is the same as the number of points in l. □

Corollary. *Every projetive plane of order n is an $S(2, n+1, n^2+n+1)$ design.*

Definition. F a fieled. $PG(2, F)$ is the projective plane over F where P is the set of 1-dimensional subspaces of F^3, L are all 2-dimensional subspaces of F^3.

In fact the Fano Plane is $PG(2, 2)$. And $PG(2, F_q)$ is a $S(2, q+1, q^2+q+1)$ design.

An incidence structure with points and lines is an affine plane if

- For any two points there is a unique line containing them
- For any points x, line l which does not contain x, then there exists a unique line l' disjoint from l containing x.
- there exist three points not all on a line.

Two lines are parallel if their intersection is trivial.

Theorem. *If $S = (P, L)$ is an affien plane, then*

- *parallel is an equivalence relation*
- *There is a number n called the order of S st, every line has size n, every point is on $n+1$ lines, $v = n^2$ and $b = n^2 + n$.*

Given a projective plane, we delete a line and all points on it to obtain an affine plane. Given an affine plane, we add point such that each line in a parallel class is incident to the same one and then add them all to new line.

Definition. Two triangles are perspective from x if the corresponding vertices lie on lines with x.

Definition. Two triangles are perspective from a line l if the points of intersection of l with the extended sides of the triangles are the same.

Theorem. *(Desorgues) in $PG(2, F)$ If two triangles are perspective from a point then they are perspective from a line.*

Proof. We may assume $a_1 =< 1, 0, 0 >, b_1 =< 0, 1, 0 >, c_1 =< 0, 0, 1 >, x =< 1, 1, 1 >$. Then $a_2 =< \alpha, 1, 1 >, b_2 =< 1, \beta, 1 >, c_2 =< 1, 1, \gamma >$. Consider the lines perpendicular to the each pair of a_1, b_2, c_1. Show that things work out. □

$PG(n, F)$ consists of the $1, 2, \ldots, n$ dimensional subspaces of F^{n+1}.

$GL(n+1, F)$ is the invertible $(n+1)x(n+1)$ matrices over F. Let $H \subset GL(n+1, F)$ be the subgroup of matrices which are multiples of I. Then H is the center of $GL(n+1, F)$.

$PGL(n+1, F) = GL(n+1, F)/H$

A projctive plane is Desarguesian if his Thm holds.

Theorem. *A finite projective plane is Desarguesian* $\iff$ *it is isomorphic to* $PG(2, F)$ *for some finite field* F*.*

Theorem. *A finite projective geometry is Desarguesian* $\iff$ *it is isomorphic to* $PG(n, F)$ *for some finite field* F*.*

Definition. If s, t are in a poset, the meet of a, b is the unique maximal element between below a, b.

Definition. If s, t are in a poset, the join of a, b is the unique minimal element between above a, b.

Definition. A poset is a lattice if any two elements have both a meet and a join.

Definition. A poset is ranked if we can assign to each element a rank such that when $a < b$ $rank(a) < rank(b)$

Definition. A ranked lattice is modular if $\forall a, b$ $rank(a \cup b) + rank(a \cap b) = rank(a) + rank(b)$.

Definition. A ranked lattice is atomic if it is equal to the join of all rank one elements it dominates.

If your lattice is atomic, every element can be described as the join of some subset of rank 1 elements.

Definition. A lattice of rank $n+1$ is connected if no two elements each of rank $< n+1$ dominate all rank 1 elements.

Definition. A projective incidence geometry is a connected atomic modular ranked lattice.

Theorem. *Every Desarguesian projective incidence geometry is isomorphic to* $PG(n, F)$ *for some* $n \geq 2$*.*

Theorem. *Every projective incidence geometry of rank* ≥ 4 *is Desarguesian.*

Proof. Let a_1, b_1, c_1 a_2, b_2, c_2 be triangles which are perspective from x. Let $P_1 = a_1 \vee b_1 \vee c_1$ and $P_2 = a_2 \vee b_2 \vee c_2$.

Consider $P_1 \neq P_2$. $P_1, P_2 \subset T = x \vee a_1 \vee b_1 \vee c_1$.

Let's assume $rank(P_1) = rank(P_2) = 3, rank(T)$. Let $L = P_1 \wedge P_2$. $a_1 \vee b_1 \wedge a_2 \vee b_2$ is a point in P_1, P_2, so is a point on L.

In the other case, we take a line through x and make a new triangle and use the previous case. □

Construction: Let G be an abelian group of order n^2 and let $H_0, \ldots H_n$ be subgroups of order n who intersect trivially with each other and each has size n. Let L be the set of all cosets of the form $g + H_i$. We claim the sets of L form an affine plane.

Construction: Let V be a 2 dimensional vector space over F_{q^2}. Note V is also a 4-dimensional vector space over F_q. Let $U_0, \ldots U_{q^2}$ be the q dimensional subspaces of V considered as a vector space over F_{q^2}.

Let W_0 be a q dimensional subspace over F_q which is not one of U_i.

3 Designs

Proposition. *If $0 \leq i \leq t$, then the number of blocks containing a fixed i element subset of points is $\lambda \frac{\binom{v-i}{t-i}}{\binom{k-i}{t-i}}$.*

Proof. Count the blocks in two ways. □

Definition. The replication number is the number of blocks containing a given point. $r = \lambda \frac{v-1}{k-1} = \frac{bk}{v}$.

Definition. If $D = (P, B)$ is a t-design and $I \subset P, |I| = i \leq t$. $D_I = (P - I, B - I : I \subset B)$. This gives a $S_\lambda(t - i, k - i, v - i)$ design. This is the derived design.

Number of sets disjoint from a set of size j is $\frac{\lambda \binom{v-j}{k}}{\binom{v-t}{k-t}}$.

If $i + j \leq t$ then the number of blocks containing a given i element set and disjoint from a j-element set, is $\frac{\lambda \binom{v-i-j}{k-i}}{\binom{v-t}{k-t}}$

Definition. The residual design D^j is the set of all blocks not containing a set of size j.

Theorem. *If $\lambda = 1$ then for any $t-$design that is not a $t+1$-design we have $v \geq (t + 1)(k - t + 1)$.*

Proof. We can find a $t+1$ element set that is not contained in a block. Each $t-$element subset is contained in a unique block. And the blocks must be otherwise unique. Thus there are at least $(t+1)(k-t)+(t+1) = (t+1)(k-t+1)$ points □

Definition. If $D = (P, B)$ is an incidence structure, the incidence matrix is indexed by points on the rows and blocks on the columns. Call the matrix N.

When $t = 2$, we have NN^T with r along the diagonol and λ elsewhere.

$sI + tJ$ has spectrum $\{tv + s, s^{v-1}\}$. So NN^T has spectrum $\{v\lambda + r - \lambda, (r-\lambda)^{v-1}\}$. $v\lambda + r - \lambda = rk$. $Det(NN^T) = rk(r-\lambda)^{v-1}$. Since the determinant is non-zero, $b \geq v$.

A design is called square if $v = b$ (symmetric).

If we have a square 2-design then $det(NN^T) = det(N)^2$. $r = k$ and so $r - \lambda$ is a square.

Theorem. *(Ray-Chandlar, Wilson) If P, B is a $t-(v,k,\lambda)$ design with $t \geq 2s, v \geq k+s$, then $b \geq \binom{v}{s}$.*

Proof. Let N_i be the matrix with i element subsets indexing the rows and the blocks indexing the columns. There is a 1 if that block contains that subset.

Let W_{ij} be the incidence matrix indexing the rows by i element subsets and the columns by all possible j element subsets. Then we claim that $N_s N_s^T = \sum_{i=0}^{s} b_{2s-i}^{i} W_{is}^T W_{is}$.

The EF entry of $N_s N_s^T$ is the number of blocks containing $E \cup F$ which is just b_{2s-l}. The EF entry of $W_{is}^T W_{is}$ is the number of i element subsets containted in E and F which is $\sum_{i}^{s} b_{2s-i}^{i} \binom{l}{i}$. □

Theorem. *Let $v, k, \lambda \in N$ satisfy $\lambda(v-1) = k(k-1)$ then a necessary condition for the existence of a square $2-(v,k,\lambda)$ design is:*

- *$k - \lambda$ is a square if v is even.*
- *$z^2 = (k-\lambda)x^2 + (-1)^{\frac{v-1}{2}}\lambda y^2$ has a nontrivial integer solution if v is odd.*

Theorem. *Every number can be written as the sum of 4 squares. A number can be written as the sum of 2 squares iff all primes that divide it an odd number of times are of the form 1 mod 4.*

Proof. Suppose a design exists. Let N be the incidence matrix. Consider the system $Nv = L$ and look at the form $L_1^2 + L_2^2 + \cdots + L_v^2$. The coefficient of $x_i x_j$ is 2λ when $i \neq j$ and is k when $i = j$. So $\sum L_i^2 = \lambda(\sum x_i)^2 + (k - \lambda)(\sum x_i^2)$. Suppose $v \equiv 1 mod 4$. We can chosoe a_1, a_2, a_3, a_4 s.t. $k - \lambda = a_1^2 + a_2^2 + a_3^2 + a_4^2$. Then $\sum L_i^2 = (\sum a_i^2)(\sum x_i^2) + \lambda(\sum x_i)^2$. We can multiply $a_i *$ four tuples of x_is. So we get $\sum L_i^2 = \sum y_i^2 + (k - \lambda)x_v^2 + \lambda(\sum x_i)^2$. For some reason we can eliminate y_1 and L_1, y_2 and L_2 etc, and we're left with where we want to be. □