Domination in Plane Triangulations

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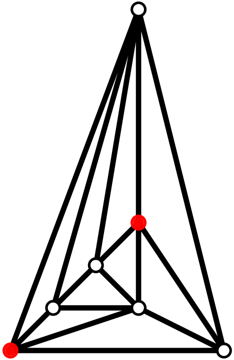
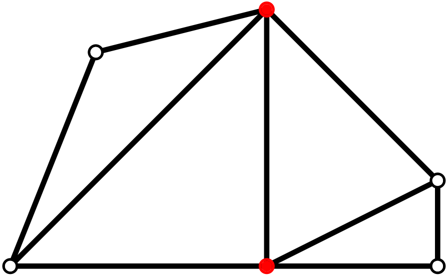
Plan

- Definitions and Examples
- Matheson and Tarjan's result
- Our new result

Definitions and Examples

- A *plane triangulation* is a graph embedded in the plane so that all faces are triangles
- A *plane near-triangulation* is a graph embedded in the plane so that at most one face is not a triangle
- A *Dominating set* in a graph G is a set $X \subseteq V(G)$ such that for each vertex $v \in V(g)$ either $v \in X$ or v has a neighbour in X.





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- If we consider the smallest of these three sets, the size must be at most $\frac{n}{3}$, so every plane triangulation has a dominating set of size at most $\frac{n}{3}$.

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- Given a plane near-triangulation of the graph G, the vertices of G can be divided into 3 sets so that each set is a dominating set and the vertices on the infinite face induce a proper colouring on that face.
- Consider first graphs that have a cut of size 1 or 2.

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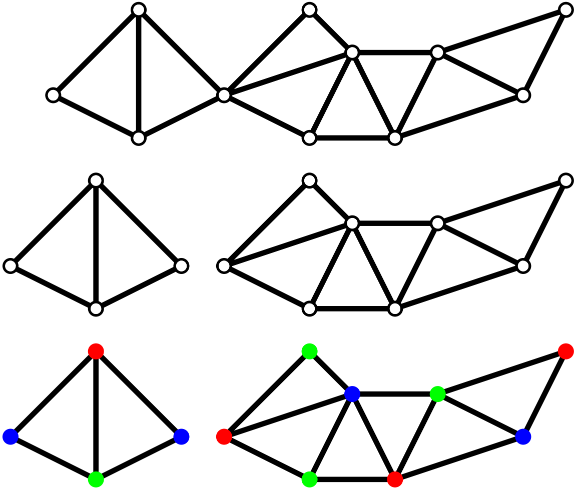

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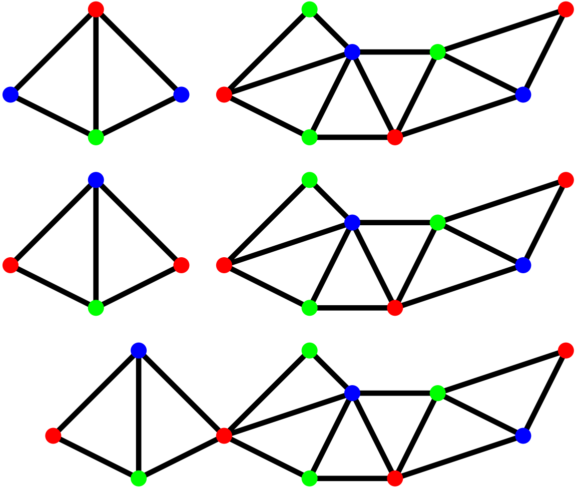

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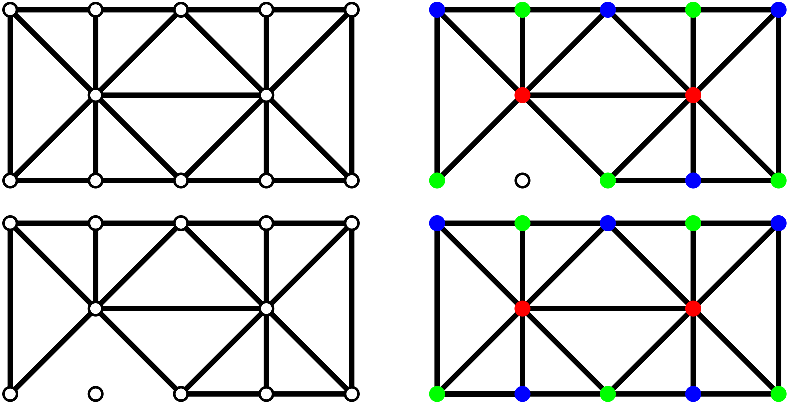

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- We can try to find more sets each of which dominate the graph.
- We can try to find a larger constant k so that we can find a dominating set of size $\frac{n}{k}$.
- In both cases, it's not possible to get better than 4. If we take a graph with m disjoint copies of K_4 and join them together to make a plane triangulation, any dominating set has size at least m.

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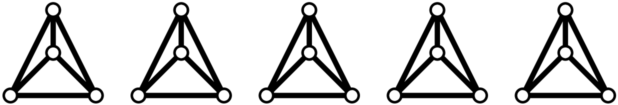

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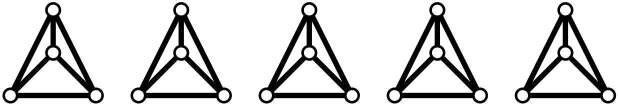

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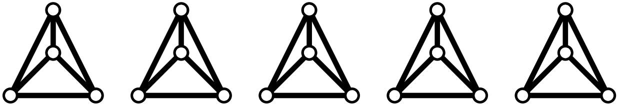

- Being able to find 4 disjoint dominating sets in every plane triangulation would imply the 4 colour theorem.
- Take a plane triangulation and put a vertex in each face joined to the three vertices of that face. We get a colouring such that each set is a dominating set. Each triangle of the original graph would have different colours on the three vertices, so we get a proper colouring with 4 colours.

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$$\frac{7f + 2p + 7n_0 + 5n_1 + 3n_2 + 2n_3 + \lambda + 2\mu}{7} = \frac{c(G)}{7}$$

where f is the number of vertices that are forced to be in the dominating set, p is the number of vertices that are "predominated," n_0 is the number of isolated vertices, n_1 is the number of vertices of degree 1, n_2 is the number of vertices of degree 2, n_3 is the number of vertices of degree > 3, μ is the number of components isomorphic to octohedron or octohedron$^-$ and λ is the number of blocks isomorphic to some elements in a list of size 4.

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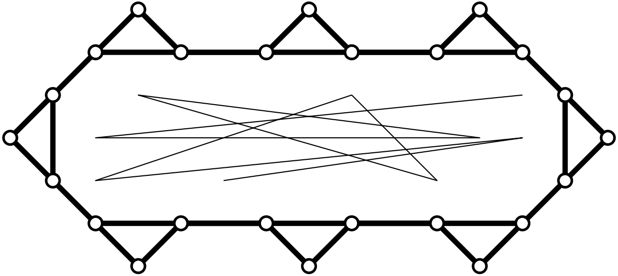

- Any graph of this type requires $\frac{n}{3}$ vertices.

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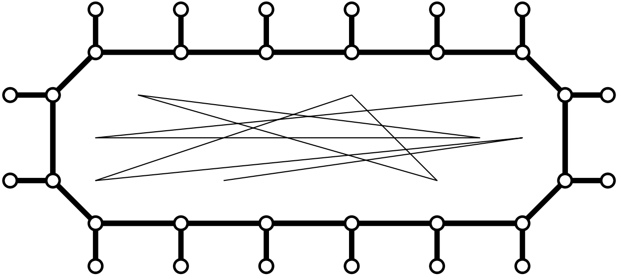
- The same problem arises for vertices of degree 1.



- Any graph of this type requires $\frac{n}{2}$ vertices.

New Result

$$\frac{7f + 2p + \mathbf{7n_0} + \mathbf{5n_1} + \mathbf{3n_2} + \mathbf{2n_3} + \lambda + 2\mu}{7}$$

- n_3 is vertices of degree > 3
- n_2 is vertices of degree 2
- n_1 is vertices of degree 1
- n_0 is vertices of degree 0

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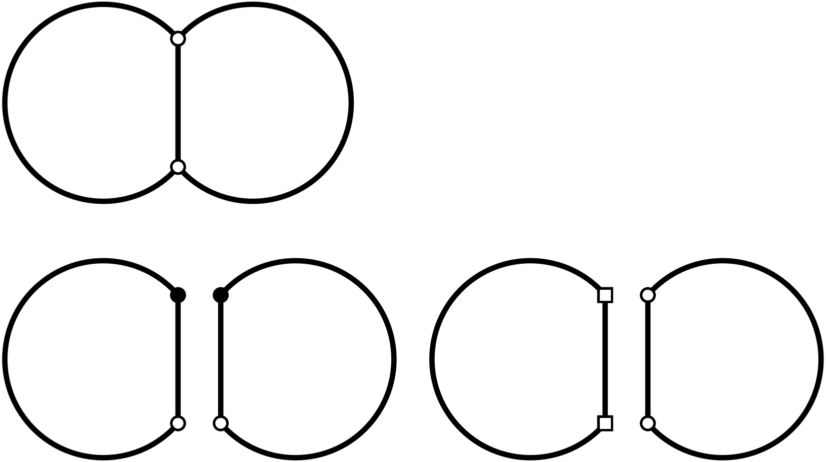

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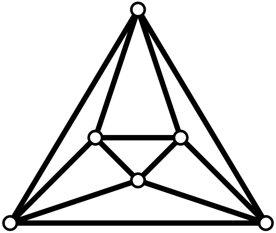
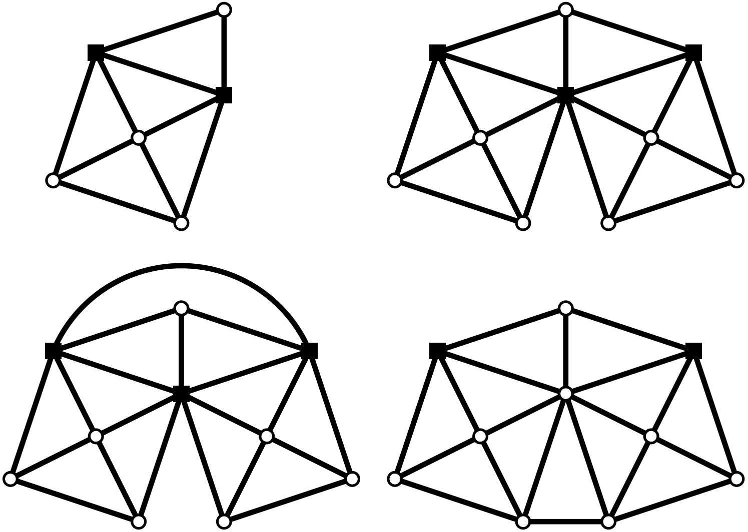


- Neither of these graphs has a dominating set consisting of only a single vertex. Any minimum size dominating set contains 2 vertices, but we are only allowed a set of size $\frac{12}{7}$ for this graph.

New Result



- Each of these graphs costs 1 more to dominate than is allowed by the vertex cost formula.

New Result

$$\frac{7f + 2p + 7n_0 + 5n_1 + 3n_2 + 2n_3 + \boldsymbol{\lambda} + \mathbf{2}\boldsymbol{\mu}}{7}$$

- These cover the cost of components or blocks that do not conform to the vertex cost formula.

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- Next we consider graphs that have a cut edge. Split the graph into two pieces with one endpoint of the edge in each component. We can bound the cost of the new graph that we get from doing this. Depending on what the cost of each piece is modulo 7 we can either force one end of the edge and predominate the other, or we can leave both alone.

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- We can repeat for a cut vertex. The actions chosen for the various values modulo 7 will change.

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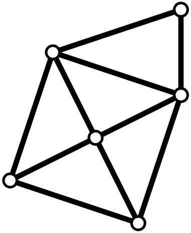

- If one side is the wheel graph, then placing a degree 2 vertex can create this, which causes problems with the induction.

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- Through case analysis we are able to reduce further to 3-connected.
- Once we're 3-connected, we start deleting edges. This allows us to assume that at least every second vertex on the infinite face has degree 3.

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- The most common way to do this is to add a vertex to the dominating set, then delete that vertex and 3 of its neighbours, creating a degree 2 vertex in the process.
- The frequency with which we use this operation would make it difficult to improve on the value 3.5 with this technique.

New Result

$$\frac{7f + 2p + 7n_0 + 5n_1 + 3n_2 + 2n_3 + \lambda + 2\mu}{7}$$

- Any plane triangulation on more than 6 vertices has no vertices of degree < 3 and is not one of our special graphs that costs more.

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- All but 2 plane triangulations have dominating sets of size $\frac{n}{3.5}$.