Domination in Plane Triangulations

Lino Demasi and Matt DeVos

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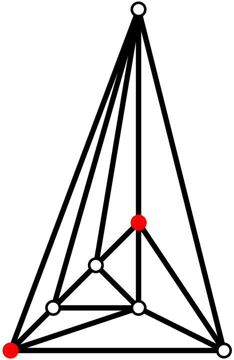
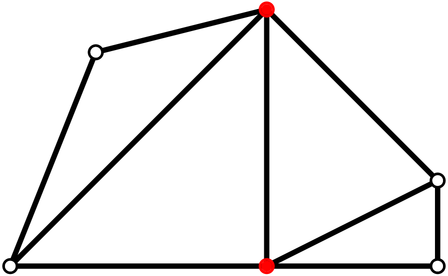
Plan

- Definitions and Examples
- Matheson and Tarjan's result
- Our new result
- Open Questions

Definitions and Examples

- A *plane triangulation* is a graph embedded in the plane so that all faces are triangles
- A *plane near-triangulation* is a graph embedded in the plane so that at most one face is not a triangle
- A *Dominating set* in a graph G is a set $X \subseteq V(G)$ such that for each vertex $v \in V(g)$ either $v \in X$ or v has a neighbour in X.





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- If we consider the smallest of these three sets, the size must be at most $\frac{n}{3}$, so every plane triangulation has a dominating set of size at most $\frac{n}{3}$.

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- Given a plane near-triangulation of the graph G, the vertices of G can be divided into 3 sets so that each set is a dominating set and the vertices on the infinite face induce a proper colouring on that face.

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- We can try to find more sets each of which dominate the graph.
- We can try to find a larger constant k so that we can find a dominating set of size $\frac{n}{k}$.
- In both cases, it's not possible to get better than 4. If we take a graph with m disjoint copies of K_4 and join them together to make a plane triangulation, any dominating set has size at least m.

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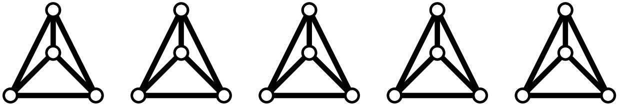

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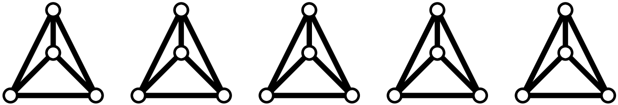

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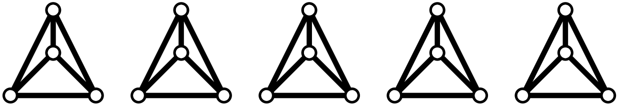

- Being able to find 4 disjoint dominating sets in every plane triangulation would imply the 4 colour theorem.
- Take a plane triangulation and put a vertex in each face joined to the three vertices of that face. We get a colouring such that each set is a dominating set. Each triangle of the original graph would have different colours on the three vertices, so we get a proper colouring with 4 colours.

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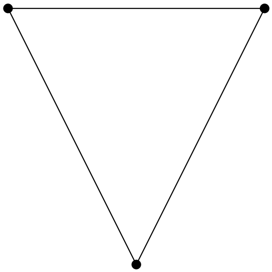

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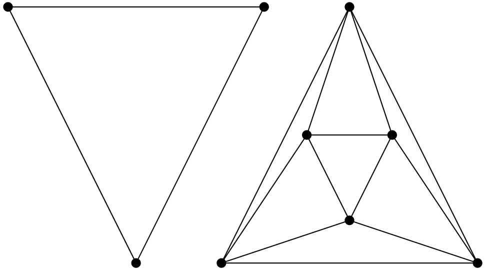

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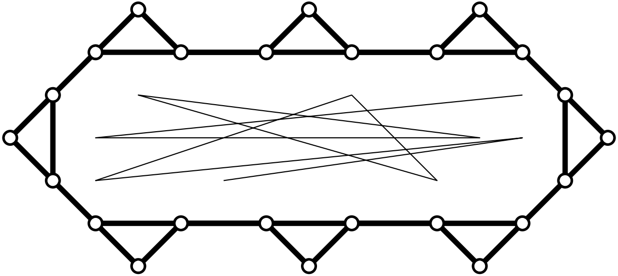

- Any graph of this type requires $\frac{n}{3}$ vertices.

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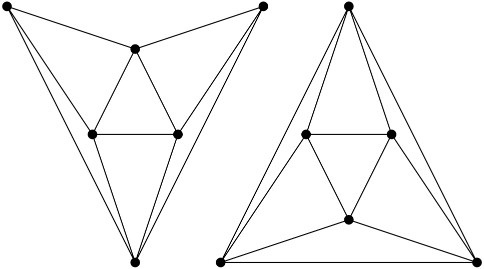

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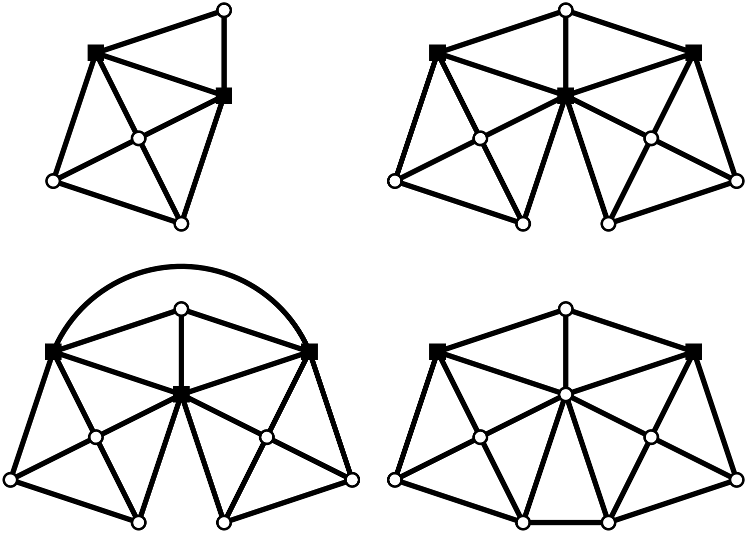

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- Need that $c(G') \leq c(G) - 7$

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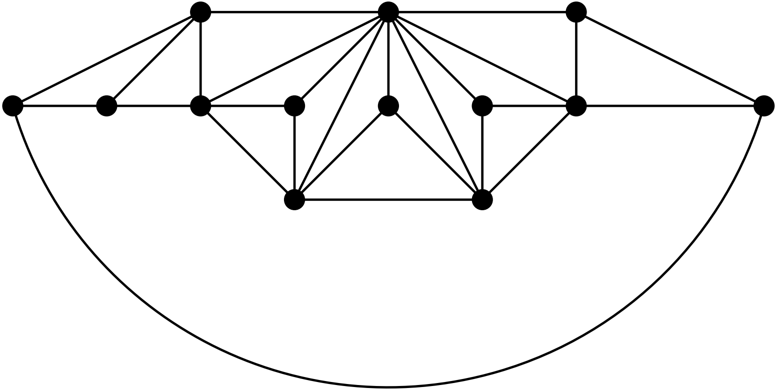

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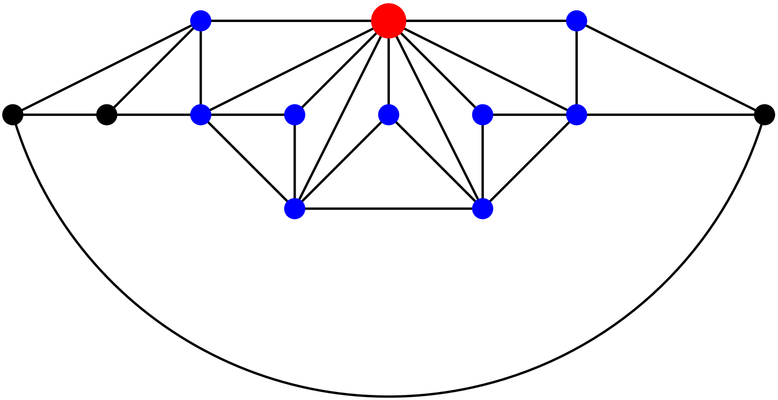

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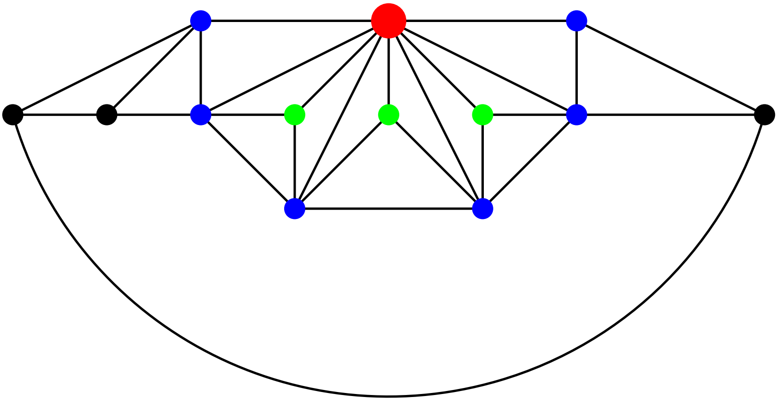

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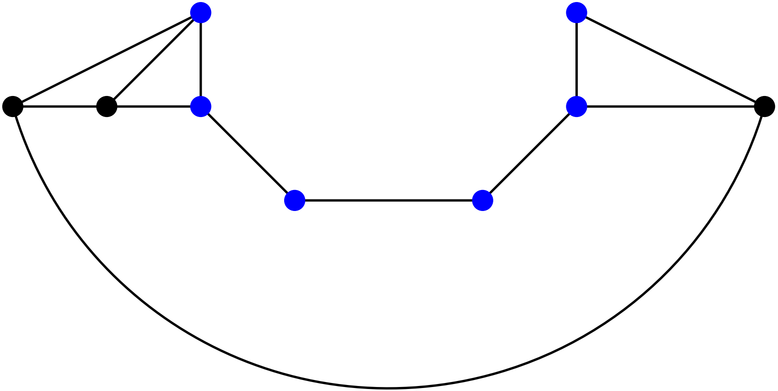

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- There are lots of different configurations the boundary can have. How do we reduce the number of possibilities to make checking them all reasonable?
- If two consecutive vertices on the boundary each have degree at least 4, we can delete the edge between them.

New Result

- **Theorem (D and DeVos):** Every plane near-triangulation G has a dominating set of size at most:

$$\frac{7f + 2p + 7n_0 + 5n_1 + 3n_2 + 2n_3 + \lambda + 2\mu}{7} = \frac{c(G)}{7}$$

where f is the number of vertices that are forced to be in the dominating set, p is the number of vertices that are "predominated," n_0 is the number of isolated vertices, n_1 is the number of vertices of degree 1, n_2 is the number of vertices of degree 2, n_3 is the number of vertices of degree > 3, μ is the number of components isomorphic to octohedron or octohedron$^-$ and λ is the number of blocks from the list of size 4.

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- When we have a triangulation that is not one of our special graphs, this reduces to $\frac{2n}{7}$.
- **Theorem (D and Devos):** All but three plane triangulations have dominating sets of size $\frac{2n}{7}$.

Open Questions

- What about triangulations on other surfaces?
- What is we do not allow vertices of degree 3?
- Can we get a dominating set of size $\frac{n}{4}$?
- Can we get 4 dominating sets?