Research Notes - Hamilton Cycle Space and Colour Switching

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1 Hailton Cycle Space

Definition. The cycle space of G is the set $CS(G) = \{C_1 \Delta C_2 \Delta \cdots \Delta C_n : C_i \in C(G)\}$ where $C(G)$ is the set of cycles of a graph and Δ is the symmetric difference.

Definition. The cycle space of G is the set $HS(G) = \{C_1 \Delta C_2 \Delta \cdots \Delta C_n : C_i \in C(G)\}$ where $C(G)$ is the set of Hamilton cycles of a graph and Δ is the symmetric difference.

For which graphs is $HS(G) = CS(G)$?

Consider graphs of the form $C_n * K_2$. These graphs are two cycles joined linearly by a perfect matching. There are two types of Hamilton Cycles in these graphs. First, we have a cyle which uses a consecutive pair of matching edges and $n-1$ arcs along each cycle. These exist for all values of n. Second, we have cycles that use all the matching edges and alternating cycle edges on each side. These exist only for n even.

For n even, if we take the symmetric difference of $n-1$ of the first type of cycle we will get a 4-cycle. If we combine a cycle of the second type with every alternating 4-cycle, we get one of the n cycles. This gives us all cycles in the generating set for the $CS(G)$, since the faces generate this set. Thus $HS(G) = CS(G)$.

For n odd, we cannot get the 4-cycle by the same construction. In fact, all of our generating cycles have an even number of arcs on each of the cycles, so any symetric difference of them must also have this property. Thus, we cannot get 4-cycles and we cannot get either of the $n-$cycles. We are able to get $6-$cycles.

Consider complete bipartite graphs K_n, n. For $n \geq 2$ this graph is Hamiltonian. Consider $1, 3$ and $2, 4$ on opposite sides of the bipartition. Then we can take cycles $P, x, 1, 2, 3, 4, y$, $P, x, 1, 4, 3, 2, y$, where x is on the same side as $2, 4$, y is on the same side as $1, 3$, and P is a Hamilton path that uses the remaining vertices. The symmetric difference of these cycles is $4, 1, 2, 3, 4$, which is a $4-$cycle. From these cycles we are able to get all cycles.

2 Colour Switching

Given a graph G and a proper $k-$colouring, is it possible to convert this to a different proper $k-$colouring by a sequence of vertex colour changes where each intermediate colouring is also proper?

For infinite trees, we require $\delta + 2$ colours. During the talk, we said we need $\Delta + 2$, but an inductive type argument will work for $\delta + 2$. If we only allow $\delta + 1$, then we can construct a $\delta + 1$ regular graph which is frozen. That is, every vertex has every other colour appearing on its neighbour set, so cannot be changed.

For finite trees however, we need only 3 colours. If we consider a leaf node, then we can remove it and colour the rest by induction. Whenever we need to alter the colour of the vertex the leaf was joined to, if it has the colour 1 and wants to be coloured with the colour of the leaf (say 2), then we give the leaf colour 3.

For any planar graph, a similar argument works with 7 colours since we must have a vertex of degree at most 5. There exist frozen colourings for $5-$regular planar graphs with 6-colours.